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QUASI-HOPF ALGEBRAS ASSOCIATED WITH $\mathfrak{sl}_2$ AND COMPLEX CURVES

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ABSTRACT

We construct quasi-Hopf algebras quantizing double extensions of the Manin pairs of Drinfeld, associated to a curve with a meromorphic differential, and the Lie algebra $\mathfrak{sl}_2$. This construction makes use of an analysis of the vertex relations for the quantum groups obtained in our earlier work, PBW-type results and computation of R-matrices for them; its key step is a factorization of the twist operator relating "conjugated" versions of these quantum groups.

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Introduction

In [6], V. Drinfeld introduced Manin pairs, attached to an absolutely simple Lie algebra over a complex curve X with a meromorphic differential ω. In the case where the genus of X is zero or one, special cases of his construction give rise to Manin triples, whose quantization are the Yangians, quantum affine algebras and elliptic algebras. Drinfeld posed the problem of quantizing these general Manin pairs, in the sense of quasi-Hopf algebras.

In this paper, we solve this problem in the untwisted case where the Lie algebra is equal to $\mathfrak{a} \otimes \mathbb{C}(X)$, $\mathfrak{a} = \mathfrak{sl}_2$, for an arbitrary curve X.

In that case, Drinfeld's Manin pair presents itself as follows. Let S be a finite set of points of X containing the zeroes and poles of ω, k_s be the local field at each $s \in S$, and $k = \bigoplus_{s\in S} k_s$. Endow $\mathfrak{a} \otimes k$ with the scalar product given by the tensor product of the Killing form of $\mathfrak{a}$ and $\langle f, g\rangle_k = \sum_{s\in S} \mathrm{res}_s(fg\omega)$. Let R be the ring of functions on X, regular outside S; it can be viewed as a subring of k. The ring R is a Lagrangian (that is, maximal isotropic) subspace of k (a proof of this fact is in the Appendix), so that $(\mathfrak{a} \otimes k, \mathfrak{a} \otimes R)$ forms a Manin pair.

In our earlier paper [8], we introduced a double extension $(\mathfrak{g}, \mathfrak{g}_R)$ of this Manin pair. The Lie algebra $\mathfrak{g}$ is a direct sum $(\mathfrak{a} \otimes k) \oplus \mathbb{C}D \oplus \mathbb{C}K$, with K a central element and D a derivation element, and $\mathfrak{g}_R$ is equal to $(\mathfrak{a} \otimes R) \oplus \mathbb{C}D$. In [8], we considered a certain Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$, obtained from this pair by a classical twist, and we constructed a quantization $U_\hbar\mathfrak{g}$ of this Manin triple.

Let us describe in more detail the Manin triple of [8]. Let $\mathfrak{a} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be a Cartan decomposition of $\mathfrak{a}$. Let Λ be a Lagrangian complement of R in k. The Lie algebra $\mathfrak{g}_+$ is then defined as $(\mathfrak{h} \otimes R) \oplus (\mathfrak{n}_+ \otimes k) \oplus \mathbb{C}D$, and $\mathfrak{g}_-$ as $(\mathfrak{h} \otimes \Lambda) \oplus (\mathfrak{n}_- \otimes k) \oplus \mathbb{C}K$.

The Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ has the following interpretation. Recall that the affine Weyl group of $\mathfrak{g}$ is the semi-direct product of a group of translations, isomorphic to $\mathbb{Z}^S$, with the Weyl group of $\mathfrak{a}^S$. The triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ can then be viewed as the limit, for the length of the Weyl group element becoming infinite, of the triple obtained from $(\mathfrak{g}, \mathfrak{g}_R, \mathfrak{g}_\Lambda)$ [with $\mathfrak{g}_\Lambda = (\mathfrak{a}\otimes\Lambda)\oplus\mathbb{C}K$] by conjugation by an affine Weyl group element corresponding to a positive translation. A similar procedure had been employed earlier by Drinfeld in [6]. We recall the results of [8] in sections 1 and 2.

Let us now describe the main points of the present work. We first construct a subalgebra $U_\hbar\mathfrak{g}_R \subset U_\hbar\mathfrak{g}$; this inclusion deforms the inclusion of Lie algebras $\mathfrak{g}_R \subset \mathfrak{g}$ (section 4). This construction is as follows. The main difficulty of [8] was to produce the correct relations for the quantum counterparts of the algebras generated by $\mathfrak{n}_\pm \otimes k$. These relations are presented in terms of generating series; they are usually called vertex relations. We observe that there exists a system of such relations, in which the products of generating series are multiplied by scalar functions belonging to $R \otimes R$. This system can be modified so as to define vertex relations for an algebra $U_\hbar\mathfrak{g}_R$ (section 4.1), which turns out to be a deformation of the enveloping algebra $U\mathfrak{g}_R$ (section 4.2). To show that this algebra is indeed a subalgebra of $U_\hbar\mathfrak{g}$, we have to establish Poincaré–Birkhoff–Witt (PBW) type results for $U_\hbar\mathfrak{g}$ (Proposition 3.2). These results follow from general similar results on algebras presented by vertex relations (section 3.1).

Let Δ denote the coproduct of $U_\hbar\mathfrak{g}$. The subalgebra $U_\hbar\mathfrak{g}_R$ then satisfies $\Delta(U_\hbar\mathfrak{g}_R) \subset U_\hbar\mathfrak{g}\hat{\otimes}U_\hbar\mathfrak{g}_R$ (see Proposition 4.4). This motivates the following construction.

Consider the Manin triple $(\mathfrak{g}, \bar{\mathfrak{g}}_+, \bar{\mathfrak{g}}_-)$, obtained from $(\mathfrak{g}, \mathfrak{g}_R, \mathfrak{g}_\Lambda)$ as the limit of conjugations by negative translations (or equivalently, from $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ by conjugation by card(S) copies of the nontrivial element of the Weyl group of $\mathfrak{a}$). Using the results of [8], it is easy to produce a quantization $(U_\hbar\bar{\mathfrak{g}}, \bar{\Delta})$ of this Manin triple. We then show that the Hopf algebras $U_\hbar\mathfrak{g}$ and $U_\hbar\bar{\mathfrak{g}}$ are isomorphic as algebras (Proposition 2.3), and their coproducts are conjugated under some $F \in U_\hbar\mathfrak{g}^{\hat{\otimes}2}$ (Proposition 5.4), which also satisfies cocycle identities (Proposition 5.3); in other words, both Hopf algebras are connected by a twist, in the sense of [6]. Let us explain how this result is obtained.

F is constructed as $\sum_i \alpha^i \otimes \alpha_i$, for $(\alpha^i), (\alpha_i)$ dual bases of the subalgebras $U_\hbar\mathfrak{n}_+$ and $U_\hbar\mathfrak{n}_-$ of $U_\hbar\mathfrak{g}$ generated by the deformations of $\mathfrak{n}_\pm \otimes k$. The universal R-matrices of $(U_\hbar\mathfrak{g}, \Delta)$ and $(U_\hbar\bar{\mathfrak{g}}, \bar{\Delta})$ are then expressed simply as the products of F and a factor $\mathcal{K}$ depending on the Cartan modes (Proposition 5.2). One checks that $\mathcal{K}$ is a twist connecting $\bar{\Delta}$ and Δ' (Lemmas 5.2 and 5.3). (Δ' is Δ composed with the permutation of factors.) Therefore, since $\mathcal{R}^{-1}$ is a twist connecting Δ' and Δ, it follows that F is a twist connecting Δ and $\bar{\Delta}$ (Propsosition 5.3 and 5.4).

On the other hand, we have $\bar{\Delta}(U_\hbar\mathfrak{g}_R) \subset U_\hbar\mathfrak{g}_R\hat{\otimes}U_\hbar\mathfrak{g}$ (Proposition 4.4).

It is then easy to see that any factorization of F of the form $F = F_2F_1$, $F_1 \in U_\hbar\mathfrak{g} \otimes U_\hbar\mathfrak{g}_R, F_2 \in U_\hbar\mathfrak{g}_R \otimes U_\hbar\mathfrak{g}$, yields an algebra morphism Δ_R from $U_\hbar\mathfrak{g}_R$ to $U_\hbar\mathfrak{g}_R^{\hat{\otimes}2}$, by the formula $\Delta_R = \mathrm{Ad}(F_1) \circ \Delta$. Moreover, the associator Φ of the quasi-Hopf algebra obtained from $U_\hbar\mathfrak{g}$ by the twist by F_1, belongs to $U_\hbar\mathfrak{g}_R^{\hat{\otimes}3}$ (Proposition 6.4). A simple argument shows that the antipode S_R of $U_\hbar\mathfrak{g}$ corresponding to this twist, preserves $U_\hbar\mathfrak{g}_R$. This shows that $(U_\hbar\mathfrak{g}_R, \Delta_R, \Phi, S_R)$ is a sub-quasi-Hopf algebra of the twist by F_1 of $(U_\hbar\mathfrak{g}, \Delta)$.

To obtain a factorization of F is thus the key point of our construction. This is achieved in section 6.1. Any possible solution (F_1, F_2) of the factorization identity is expressed simply in terms of left and right $U_\hbar\mathfrak{g}_R$-module maps Π, Π' from $U_\hbar\mathfrak{g}$ to $U_\hbar\mathfrak{g}_R$ applied to F, and of variable elements of $U_\hbar\mathfrak{g}_R^{\otimes 2}$. The difficulty is to show that for some choice of those elements, the factorization identity is satisfied. This is equivalent to showing that $F^{-1}[(\Pi \otimes 1)F]$ and $[(1 \otimes \Pi')F]F^{-1}$ belong to $U_\hbar\mathfrak{g} \otimes U_\hbar\mathfrak{g}_R$, resp. to $U_\hbar\mathfrak{g}_R \otimes U_\hbar\mathfrak{g}$. For this, we use the pairing between the quantizations $U_\hbar\mathfrak{g}_+$ and $U_\hbar\mathfrak{g}_-$ of $\mathfrak{g}_+$ and $\mathfrak{g}_-$, and the computation of the orthogonals of their intersections with $U_\hbar\mathfrak{g}_R$ (Proposition 5.5).

We close the paper by some remarks related to our construction. We observe that the quasi-Hopf algebras $U_\hbar\mathfrak{g}$ and $U_\hbar\mathfrak{g}_R$ fit in an inductive system w.r.t. the relation $S \subset S'$, and that the corresponding inductive limit is a quantization of double extensions of the adelic versions of Drinfeld's Manin pairs (section 6.3). We also find an algebra automorphism of $U_\hbar\mathfrak{g}$, deforming the action of the generator of the Weyl group of $\mathfrak{a}$ (section 6.4). Section 7 is devoted to analogues and generalizations of $U_\hbar\mathfrak{g}$. In 7.1, we exploit the fact that the central terms occur only in the exponential form $\exp(\hbar K\partial)$ [where ∂ is the derivation of k defined by $\partial f = df/\omega$] to construct analogues of $U_\hbar\mathfrak{g}$, where $\exp(\hbar K\partial)$ is replaced by a more general automorphism of k. In 7.2, we construct analogues of those algebras and of their Weyl group automorphism, associated with discrete sets.

An expression of F was given in an earlier version of this work. However, this expression is not correct, as it was pointed out in [4]. In Remark 7.4, we discuss this problem and how this modifies the proofs of the results of [8, 9], which remain valid.

Let us now mention some possible extensions of this work. It is natural to ask how the algebras introduced here depend on the pair (X, ω). The algebra $U_\hbar\mathfrak{g}$ probably possesses level 1 modules similar to those studied in the Yangian and quantum affine cases. It would then be interesting to study quantum Knizhnik-Zamolodchikov type equations for traces of corresponding intertwining operators. Another subject of interest could be the representation theory of $U_\hbar\mathfrak{g}_R$. In [8], we studied level zero representations of $U_\hbar\mathfrak{g}$, indexed by formal discs around each point of S; these representations are also $U_\hbar\mathfrak{g}_R$-modules, and as such their parameter could probably take values outside those discs.

Finally, the question arises whether the formulas defining $U_\hbar\mathfrak{g}$ can be written is closed form (rather than in the sense of formal series) and can be analytically continued to complex values of $\hbar$. In general, the solution to this problem might be related to functional equations satisfied by the structure constants of this algebra. Let us however mention two cases where the answer to this question is positive. One of them is when X is an elliptic curve, and $\omega = dz$. This case was treated by G. Felder and one of us ([8]). We showed, using arguments of the present paper, the connection of the algebra $U_\hbar\mathfrak{g}$ with the elliptic quantum groups of [12]. The other case was treated in [11]. There we study the case of a genus > 1 curve X, with differential ω regular and having only double poles. In that case, the structure constants of $U_\hbar\mathfrak{g}$ involve some theta-functions and odd theta-characteristics of X. Also let us mention the work [3], where "analytic" algebras with close analogy to $U_\hbar\mathfrak{g}$ were introduced.

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1. Manin pairs and triples

1.1 COMPLETED TENSOR PRODUCTS AND ALGEBRAS. Let V, W be complex Tate's vector spaces (see [2] for the definition; the only examples of Tate's vector spaces we will use are either discrete or isomorphic to the sum of a finite number of copies of a field of Laurent power series in one variable).

The completed tensor product of V and W is defined as the inverse limit

$$\lim_{\leftarrow a,b}(V \otimes W/V_a \otimes W_b),$$

V_a, W_b being a system of vector spaces that consitute neighborhoods of zero in V, W, and denoted by $V\hat{\otimes}W$. For example, with $V = \mathbb{C}((v))$ and $W = \mathbb{C}((w))$ endowed with the v- and w-adic topologies, we have $V\hat{\otimes}W = \mathbb{C}[[v,w]][v^{-1}, w^{-1}]$ which we will denote as $\mathbb{C}((v,w))$.

With the same notation, the completed tensor algebra of V is defined as

$$\bigoplus_{i\geq 0} V^{\hat{\otimes} n},$$

and endowed with the obvious product, and denoted by $T(V)\hat{}$. These objects are independent of the basis of neighborhoods chosen.

$V\hat{\otimes}W$ (resp. $T(V)\hat{}$) is a separated complete topological vector space (resp. algebra), with a basis of neighborhoods of zero given by

$$\lim_{\leftarrow a,b}(V_n \otimes W_m/V_a \otimes W_b)$$

(resp. the subalgebras generated by the V_n).

We will also define $V\bar{\otimes}W$ as the inverse limit

$$\lim_{\leftarrow a,b}(V\otimes W)/(V_a\otimes W+V\otimes W_b),$$

whose topology is defined by the basis of neighborhoods of zero given by

$$\lim_{\leftarrow a,b}(V_n\otimes W+V\otimes W_m)/(V_a\otimes W+V\otimes W_b).$$

1.2 MANIN PAIRS AND TRIPLES. Let X be a smooth, connected, compact complex curve, and ω be a nonzero meromorphic differential on X. Let S be a finite set of points of X, containing the set S_0 of its zeros and poles. For each $s\in S$, let k_s be the local field at s and

$$k=\bigoplus_{s\in S}k_s.$$

Let R be the ring of meromorphic functions on X, regular outside S; R can be viewed as a subring of k. R is endowed with the discrete topology and k with its usual (formal series) topology. Let us define on k the bilinear form

$$\langle f,g\rangle_k=\sum_{s\in S}\mathrm{res}_s(fg\omega),$$

and the derivation

$$\partial f=df/\omega.$$

We will use the notation $\mathfrak{r}(A)=\mathfrak{r}\otimes A$, for any ring A over $\mathbb{C}$ and complex Lie algebra $\mathfrak{r}$.

Let $\mathfrak{a}=\mathfrak{sl}_2(\mathbb{C})$. Define on $\mathfrak{a}(k)$ the bilinear form $\langle,\rangle_{\mathfrak{a}(k)}$ by

$$\langle x\otimes\epsilon,y\otimes\eta\rangle_{\mathfrak{a}(k)}=\langle x,y\rangle_{\mathfrak{a}}\langle\epsilon,\eta\rangle_k$$

for $x,y\in\mathfrak{a},\epsilon,\eta\in k$, $\langle,\rangle_{\mathfrak{a}}$ being the Killing form of $\mathfrak{a}$, the derivation $\partial_{\mathfrak{a}(k)}$ by $\partial_{\mathfrak{a}(k)}(x\otimes\epsilon)=x\otimes\partial\epsilon$, for $x\in\mathfrak{a},\epsilon\in k$, and the cocycle

$$c(\xi,\eta)=\langle\xi,\partial_{\mathfrak{a}(k)}\eta\rangle_{\mathfrak{a}(k)}.$$

Let $\hat{\mathfrak{g}}$ be the central extension of $\mathfrak{a}(k)$ by this cocycle. We then have

$$\hat{\mathfrak{g}}=\mathfrak{a}(k)\oplus\mathbb{C}K,$$

with bracket such that K is central, and $[\xi,\eta]=([\bar{\xi},\bar{\eta}],c(\bar{\xi},\bar{\eta})K)$, for any $\xi,\eta\in\hat{\mathfrak{g}}$ with first components $\bar{\xi},\bar{\eta}$.

Let us denote by $\partial_{\hat{\mathfrak{g}}}$ the derivation of $\hat{\mathfrak{g}}$ defined by $\partial_{\hat{\mathfrak{g}}}(\xi, 0) = (\partial_{\mathfrak{a}(k)}\xi, 0)$ and $\partial_{\hat{\mathfrak{g}}}(K) = 0$.

Let $\mathfrak{g}$ be the skew product of $\hat{\mathfrak{g}}$ with $\partial_{\hat{\mathfrak{g}}}$. We have

$$\mathfrak{g} = \hat{\mathfrak{g}} \oplus \mathbb{C}D,$$

with bracket such that $\hat{\mathfrak{g}} \to \mathfrak{g}$, $\xi \mapsto (\xi, 0)$ is a Lie algebra morphism, and $[D, (\xi, 0)] = (\partial_{\hat{\mathfrak{g}}}(\xi), 0)$ for $\xi \in \hat{\mathfrak{g}}$.

View $\mathfrak{a}(k)$ as a subspace of $\mathfrak{g} = \hat{\mathfrak{g}} \oplus \mathbb{C}D = \mathfrak{a}(k) \oplus \mathbb{C}K \oplus \mathbb{C}D$, by $\xi \mapsto (\xi, 0, 0)$. Define on $\mathfrak{g}$ the pairing $\langle , \rangle_{\mathfrak{g}}$ by $\langle K, D\rangle_{\mathfrak{g}} = 1$, $\langle K, \mathfrak{a}(k)\rangle_{\mathfrak{g}} = \langle D, \mathfrak{a}(k)\rangle_{\mathfrak{g}} = 0$, $\langle \xi, \eta\rangle_{\mathfrak{g}} = \langle \xi, \eta\rangle_{\mathfrak{a}(k)}$ for $\xi, \eta \in \mathfrak{a}(k)$.

Endow $\mathfrak{a}(k)$ with $\langle , \rangle_{\mathfrak{a}(k)}$. The subspace $\mathfrak{a}(R) \subset \mathfrak{a}(k)$ is a maximal isotropic subalgebra of $\mathfrak{a}(k)$, as follows from Lemma 7.2. Drinfeld's Manin pair is $(\mathfrak{a}(k), \mathfrak{a}(R))$ (see [6]). In [8], we introduced the following extension of this pair. Let $\mathfrak{g}_R = \mathfrak{a}(R) \oplus \mathbb{C}D$; $\mathfrak{g}_R \subset \mathfrak{g}$ is a maximal isotropic subalgebra of $\mathfrak{g}$. The extended Drinfeld's Manin pair of [8] is then $(\mathfrak{g}, \mathfrak{g}_R)$.

In [8], we also introduced the following Manin triple. Let Λ be a Lagrangian complement to R in k, commensurable with $\bigoplus_{s\in S} \mathcal{O}_s$ (where $\mathcal{O}_s$ is the completed local ring at s). Let $\mathfrak{n}_+ = \mathbb{C}e$, $\mathfrak{n}_- = \mathbb{C}f$, $\mathfrak{h} = \mathbb{C}h$. Let

$$\mathfrak{g}_+ = \mathfrak{h}(R) \oplus \mathfrak{n}_+(k) \oplus \mathbb{C}D, \quad \mathfrak{g}_- = (\mathfrak{h} \otimes \Lambda) \oplus \mathfrak{n}_-(k) \oplus \mathbb{C}K;$$

then $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, and both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are maximal isotropic subalgebras of $\mathfrak{g}$. The Manin triple is then $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$.

We will also consider the following Manin triple, which we may consider as being obtained from the previous one by the action of the nontrivial element of the Weyl group of $\mathfrak{a}$. Let

$$\bar{\mathfrak{g}}_+ = \mathfrak{h}(R) \oplus \mathfrak{n}_-(k) \oplus \mathbb{C}D, \quad \bar{\mathfrak{g}}_- = (\mathfrak{h} \otimes \Lambda) \oplus \mathfrak{n}_+(k) \oplus \mathbb{C}K;$$

then $(\mathfrak{g}, \bar{\mathfrak{g}}_+, \bar{\mathfrak{g}}_-)$ again forms a Manin triple.

Remark 1.1 (Generalizations): It is straightforward to generalize the centerless versions of the Manin pairs and triples introduced above, as well as (as we will see) of their quantizations, to the case of a Frobenius algebra (i.e. a commutative ring k_0 with a linear form $\theta \in (k_0)^*$, such that $(a, b) \mapsto \langle a, b\rangle_{k_0} = \theta(ab)$ is a nondegenerate inner product), with a maximal isotropic subalgebra R_0.

It is also easy to generalize the Manin pairs and triples $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-), (\mathfrak{g}, \bar{\mathfrak{g}}_+, \bar{\mathfrak{g}}_-)$ and $(\mathfrak{g}, \mathfrak{g}_R)$, as well as their quantizations in the sense of formal series, to the case where the Frobenius algebra is endowed with a derivation ∂_0, such that $\theta \circ \partial_0 = 0$. ∎

1.3 CLASSICAL TWISTS. According to [4], to each of the Manin triples $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ and $(\mathfrak{g}, \bar{\mathfrak{g}}_+, \bar{\mathfrak{g}}_-)$ is associated a Lie bialgebra structure on $\mathfrak{g}$; denote by $\delta, \bar\delta : \mathfrak{g} \to \mathfrak{g}\hat\otimes\mathfrak{g}$ the corresponding cocycle maps.

Let $\mathfrak{g}_\Lambda = (\mathfrak{a} \otimes \Lambda) \oplus \mathbb{C}K \subset \mathfrak{g}$; $\mathfrak{g}_\Lambda$ is a Lagrangian complement of $\mathfrak{g}_R$ in $\mathfrak{g}$. It induces a Lie quasi-bialgebra structure on $\mathfrak{g}_R$, and from [1] follows also that there is a Lie quasi-bialgebra structure on $\mathfrak{g}$, associated to the Manin pair $(\mathfrak{g}, \mathfrak{g}_R)$ and to $\mathfrak{g}_\Lambda$; we denote by $\delta_R : \mathfrak{g} \to \mathfrak{g}\hat\otimes\mathfrak{g}$ the corresponding cocycle map.

These Lie (quasi-)bialgebra structures on $\mathfrak{g}$ are related by the following classical twist operations.

Let $(e^i)_{i\in\mathbb{N}}, (e_i)_{i\in\mathbb{N}}$ be dual bases of R and Λ; we choose them is such a way that e_i tends to 0 when i tends to ∞. Let $\epsilon^i, \epsilon_i, i \in \mathbb{Z}$ be dual bases of k, defined by $\epsilon_i = e_i$, $\epsilon^i = e^i$, $i \geq 0$; $\epsilon_i = e^{-i-1}$, $\epsilon^i = e_{-i-1}, i < 0$.

LEMMA 1.1: *Let $f = \sum_{i\in\mathbb{Z}} e[\epsilon^i] \otimes f[\epsilon_i]$; $f = f_1 + f_2$, with*

$$f_1 = \sum_{i\in\mathbb{N}} e[e_i] \otimes f[e^i],$$

and

$$f_2 = \sum_{i\in\mathbb{N}} e[e^i] \otimes f[e_i].$$

For $\xi \in \mathfrak{g}$, we have

$$\delta_R(\xi) = \delta(\xi) + [f_1, \xi \otimes 1 + 1 \otimes \xi], \quad \bar\delta(\xi) = \delta_R(\xi) + [f_2, \xi \otimes 1 + 1 \otimes \xi].$$

Proof: This is a consequence of the fact that the cocycle maps $\delta, \bar\delta, \delta_R$ are respectively equal to

$$\xi \mapsto [\sum_j \lambda_j \otimes \mu^j, \xi \otimes 1 + 1 \otimes \xi], \quad \xi \mapsto [\sum_j \lambda'_j \otimes \mu'^j, \xi \otimes 1 + 1 \otimes \xi],$$
$$\xi \mapsto [\sum_j \lambda_j^{(R)} \otimes \mu^{j(R)}, \xi \otimes 1 + 1 \otimes \xi],$$

where $(\lambda_j), (\mu^j)$, resp. $(\lambda'_j), (\mu'^j)$; $(\lambda_j^{(R)}), (\mu^{j(R)})$ are dual bases of $\mathfrak{g}_+, \mathfrak{g}_-$, resp. $\mathfrak{g}'_+, \mathfrak{g}'_-$; $\mathfrak{g}_R, \mathfrak{g}_\Lambda$. ∎

2. Quantization of Manin triples

2.1 RESULTS ON KERNELS. Recall that we have introduced dual bases $(e^i)_{i\in\mathbb{N}}, (e_i)_{i\in\mathbb{N}}$ of R and Λ. Let $a_0 \in R\hat\otimes\Lambda$ be equal to

$$a_0 = \sum_i e^i \otimes e_i.$$

Note that $R\hat{\otimes}k$ is an algebra, to which belongs a_0. Let

$$\gamma = (\partial \otimes 1)a_0 - (a_0)^2;$$

then

LEMMA 2.1 (see [8]): *γ belongs to $R \otimes R$.*

Let $\hbar$ be a formal variable and let $T : k[[\hbar]] \to k[[\hbar]]$ be the operator equal to

$$T = \frac{\mathrm{sh}(\hbar\partial)}{\hbar\partial}.$$

Let us use the notation $x \mapsto \widetilde{x}$ for the operation of exchanging the two factors of $(k\hat{\otimes}k)[[\hbar]]$.

PROPOSITION 2.1 (see [8] and [11], Proposition 1.11): *For certain elements $\phi \in (R \otimes R)[[\hbar]]$, $\psi_+, \psi_- \in \hbar(R \otimes R)[[\hbar]]$ depending on $\gamma, \widetilde{\gamma}$ and their derivatives by universal formulas, we have the following identities in $(R\hat{\otimes}k)[[\hbar]]$:*

$$\sum_i Te^i \otimes e_i = \phi + \frac{1}{2\hbar} \ln \frac{1 + a_0\psi_-}{1 + a_0\psi_+}, \quad \sum_i e^i \otimes Te_i = -\widetilde{\phi} + \frac{1}{2\hbar} \ln \frac{1 - a_0\widetilde{\psi}_+}{1 - a_0\widetilde{\psi}_-}.$$

LEMMA 2.2 (see [8]): *The expression $\sum_i Te^i \otimes e_i - e^i \otimes Te_i$ belongs to $S^2(R)[[\hbar]]$. We will denote by τ any element of $(R \otimes R)[[\hbar]]$, such that*

$$\tau + \widetilde{\tau} = \sum_i Te^i \otimes e_i - e^i \otimes Te_i, \tag{1}$$

and define the linear map $U : \Lambda[[\hbar]] \to R[[\hbar]]$ by

$$U\lambda = \langle \tau, 1 \otimes \lambda \rangle. \tag{2}$$

Note that $\sum_i Te^i \otimes e_i$ is well-defined in $(R\hat{\otimes}k)[[\hbar]]$, because e_i tends to zero as i tends to infinity. Since ∂ is a continuous map from k to itself, the same is true for the sequence $\partial^k e_i$. So $\sum_i e^i \otimes Te_i$ is well-defined in the same space; $\sum_{i\in\mathbb{Z}} T\epsilon^i \otimes \epsilon_i - \epsilon^i \otimes T\epsilon_i$ is well-defined in $(k\hat{\otimes}k)[[\hbar]]$ for the same reasons.

Remark 2.1 (see [8]): Let $q = e^\hbar$, and f be the function defined by

$$f(x) = \frac{q^\partial - 1}{\hbar\partial} \quad \text{and} \quad \tau_0 = \sum_i f(\partial)e^i \otimes e_i - e^i \otimes f(-\partial)e_i$$

(since ∂ is a continuous map from k to itself, each $\sum_i e^i \otimes \partial^k e_i$ is well-defined in $R\hat{\otimes}k$, so that τ_0 is well-defined in $(R\hat{\otimes}k)[[\hbar]]$).

Note that the formal series $f(\partial_z) - f(-\partial_w)$ is divisible by $\partial_z + \partial_w$ in $\mathbb{C}[\partial_z, \partial_w][[\hbar]]$, and denote their ratio by

$$\frac{f(\partial_z) - f(-\partial_w)}{\partial_z + \partial_w}.$$

Attach indices z and w to the first and second factors of $(k\hat{\otimes}k)[[\hbar]]$. We have

$$\tau_0 = \frac{f(\partial_z) - f(-\partial_w)}{\partial_z + \partial_w}(\gamma - \widetilde{\gamma}),$$

and τ_0 satisfies the identity (1). ∎

Consider now the quantity $\exp(2\hbar \sum_{i\in\mathbb{N}} e^i \otimes (T+U)e_i)$; it belongs to $(R\hat{\otimes}k)[[\hbar]]$. Let, for each $s \in S$, z_s be a local coordinate on X near s.

LEMMA 2.3: *For any $\alpha \in k$, we have*

$$(3) \qquad \begin{aligned} &(\alpha\otimes 1 - 1\otimes\alpha + \psi_-(\alpha\otimes 1 - 1\otimes\alpha)a_0)q^{2\sum_i (T+U)e_i\otimes e^i} \\ &= (\alpha\otimes 1 - 1\otimes\alpha + \psi_+(\alpha\otimes 1 - 1\otimes\alpha)a_0)q^{2(\tau-\phi)}. \end{aligned}$$

For any $\alpha \in k$, $(\alpha\otimes 1 - 1\otimes\alpha)a_0$ belongs to $\prod_{s,t\in S}\mathbb{C}((z_s, w_t))$. If α belongs to R, then $(\alpha\otimes 1 - 1\otimes\alpha)a_0$ belongs to $R\otimes R$.

Proof: The first part of the lemma is proved using that $\tau = \sum_{i\in\mathbb{N}} Ue_i\otimes e^i$, the second identity of Proposition 2.1 and

$$(4) \qquad (\alpha\otimes 1 - 1\otimes\alpha)\widetilde{a}_0 = -(\alpha\otimes 1 - 1\otimes\alpha)a_0, \quad \forall\alpha\in k$$

((4) follows from the fact that $a_0 + \widetilde{a}_0$ verifies $\langle a_0 + \widetilde{a}_0, \mathrm{Id}\otimes\beta\rangle_k = \beta$ for any $\beta\in k$; the product is taken with the second components of a decomposition of $a_0 + \widetilde{a}_0$).

Let us pass to the second part of the lemma. For N integer, set

$$k_N = \prod_{s\in S} z_s^{-N}\mathbb{C}[[z_s]].$$

For any $\alpha\in k$, there exists an integer N_α such that $(\alpha\otimes 1 - 1\otimes\alpha)a_0$ belongs to $k\hat{\otimes}k_{N_\alpha}$. Since $(\alpha\otimes 1 - 1\otimes\alpha)\widetilde{a}_0$ belongs to $k_{N_\alpha}\bar{\otimes}k$, and using (4), we obtain that $(\alpha\otimes 1 - 1\otimes\alpha)\widetilde{a}_0$ belongs to $(k\bar{\otimes}k_{N_\alpha})\cap(k_{N_\alpha}\bar{\otimes}k)\subset\prod_{s,t\in S}\mathbb{C}((z_s, w_t))$.

The third part of the lemma follows from the following statement. Let $\alpha\in R$; then for any $\beta\in R$, one checks that

$$\langle(\alpha\otimes 1 - 1\otimes\alpha)a_0, \beta\otimes\mathrm{Id}\rangle_k = \langle(\alpha\otimes 1 - 1\otimes\alpha)a_0, \mathrm{Id}\otimes\beta\rangle_k = 0,$$

where the products are taken with the first and second components of a decomposition of $(\alpha\otimes 1 - 1\otimes\alpha)a_0$. ∎

Remark 2.2: Attach indices z and w to the first and second factors of $k\bar{\otimes}k$. Let P be a differential operator, acting on k. The quantity $K_P(z,w) = \sum_{i\in\mathbb{Z}} P\epsilon^i \otimes \epsilon_i = \sum_{i\in\mathbb{Z}}(P\epsilon^i)(z)\epsilon_i(w)$ can be considered as a kernel for the operator P, because of the identity

$$(Pf)(w) = \mathrm{res}_{w\in S} K_P(z,w)f(w)\omega_w, \quad \forall f \in k.$$

Suppose that P preserves R. Then

$$\bar{K}_P(z,w) = \sum_{i\in\mathbb{Z}} Pe^i \otimes e_i = \sum_{i\in\mathbb{Z}}(Pe^i)(z)e_i(w)$$

can also be considered as a kernel for P, because

$$(Pf)(w) = \mathrm{res}_{w\in S}\bar{K}_P(z,w)f(w)\omega_w, \quad \forall f \in R. \qquad \blacksquare$$

Remark 2.3: The map $R \mapsto R\otimes R$, $\alpha \mapsto (\alpha\otimes 1 - 1\otimes\alpha)a_0$ defines a nontrivial element of the Hochschild cohomology $H^1(R, R\otimes R)$ (where $R\otimes R$ is the R-bimodule, with left action by multiplication on the first factor on the tensor product and right action by multiplication on the second one).

Note that there is a natural map $H^1(R, R\otimes R) \to H^1(k, k\hat{\otimes}_R(R\otimes R)\hat{\otimes}_R k)$; $k\hat{\otimes}_R(R\otimes R)\hat{\otimes}_R k$ is equal to $\prod_{s,t\in S}\mathbb{C}((z_s,w_t))$, so the image by this map of the above class is nonzero [since $a_0 \notin \prod_{s,t\in S}\mathbb{C}((z_s,w_t))$].

2.2 Presentation of $U_\hbar\mathfrak{g}$. In [8], we introduced a Hopf algebra $U_\hbar\mathfrak{g}$ quantizing $(\mathfrak{g},\delta)$.

It is the quotient of $T(\mathfrak{g})[[\hbar]]$ by the following relations. Let e, f, h be the Chevalley basis of $\mathfrak{sl}_2(\mathbb{C})$. Denote, in $T(\mathfrak{g})[[\hbar]]$, the element $x\otimes\epsilon \in \mathfrak{a}(k) \subset \mathfrak{g}$ of $\mathfrak{g}$ by $x[\epsilon]$ and let, for $r \in R$, $h^+[r] = h[r]$, $h^-[\lambda] = h[\lambda]$. Introduce the generating series

$$e(z) = \sum_{i\in\mathbb{Z}} e[\epsilon_i]\epsilon^i(z), \quad f(z) = \sum_{i\in\mathbb{Z}} f[\epsilon_i]\epsilon^i(z),$$
$$h^+(z) = \sum_{i\in\mathbb{N}} h^+[e^i]e_i(z), \quad h^-(z) = \sum_{i\in\mathbb{N}} h^-[e_i]e^i(z).$$

The relations for $U_\hbar\mathfrak{g}$ are (Fourier modes of)

$$[h^+[r], h^+[r']] = 0, \tag{5}$$

$$[K,\ \text{anything}] = 0, \quad [h^+[r], h^-[\lambda]] = \frac{2}{\hbar}\langle(1-q^{-K\partial})r, \lambda\rangle, \tag{6}$$

$$[h^-[\lambda], h^-[\lambda']] = \tag{7}$$
$$\frac{2}{\hbar}\left(\langle T((q^{K\partial}\lambda)_R), q^{K\partial}\ \lambda'\rangle + \langle U\lambda, \lambda'\rangle - \langle U((q^{K\partial}\lambda)_\Lambda), q^{K\partial}\lambda'\rangle\right),$$

(8) $[h^+[r], e(w)] = 2r(w)e(w), \quad [h^-[\lambda], e(w)] = 2[(T+U)(q^{K\partial}\lambda)_\Lambda](w)e(w),$

(9) $[h^+[r], f(w)] = -2r(w)f(w), \quad [h^-[\lambda], f(w)] = -2[(T+U)\lambda](w)f(w),$

$$[z_s - w_s + \psi_-(z,w)a_0(z,w)(z_s - w_s)]e(z)e(w)$$
$$= q^{2(\tau-\phi)(z,w)}[z_s - w_s + \psi_+(z,w)a_0(z,w)(z_s - w_s)]e(w)e(z), \quad \forall s \in S, \tag{10}$$

$$q^{2(\tau-\phi)(z,w)}[z_s - w_s + \psi_+(z,w)a_0(z,w)(z_s - w_s)]f(z)f(w) \tag{11}$$
$$= [z_s - w_s + \psi_-(z,w)a_0(z,w)(z_s - w_s)]f(w)f(z), \quad \forall s \in S,$$

$$[e(z), f(w)] = \frac{1}{\hbar}[\delta(z,w)q^{((T+U)h^+)(z)} - (q^{-K\partial_w}\delta(z,w))q^{-h^-(w)}], \tag{12}$$

$$[D, h^+[r]] = h^+[\partial r],$$
$$[D, h^-(z)] = -(\partial h^-)(z) - \sum_{i\in\mathbb{N}}[(1+q^{-K\partial})Ae_i](z)h^+[e^i] + \mathcal{A}(z), \tag{13}$$

$$[D, x^\pm(z)] = -(\partial x^\pm)(z) + \hbar\sum_{i\in\mathbb{N}}(Ae_i)(z)h^+[e^i]x^\pm(z), \tag{14}$$

for any $r, r' \in R$, $\lambda, \lambda' \in \Lambda$, $x^\pm = e, f$, with $A : \Lambda[[\hbar]] \to R[[\hbar]]$ defined by

$$A\lambda = T((\partial\lambda)_R) + \partial(U\lambda) - U((\partial\lambda)_\Lambda), \quad \forall \lambda \in \Lambda, \tag{15}$$

and

$$\mathcal{A}(z) = \sum_{i\in\mathbb{N}}[(1+q^{-K\partial})Ae_i](z)[(1-q^{-K\partial})e^i](z) \tag{16}$$
$$+ \frac{2}{\hbar}[(q^{-K\partial} - 1)Ae_i](z)e^i(z).$$

Note that A is anti-self-adjoint, so that $\sum_{i\in\mathbb{N}} Ae^i \otimes e_i = -\sum_{i\in\mathbb{N}} e^i \otimes Ae_i$, and $\sum_{i\in\mathbb{N}}(Ae_i)(z)e^i(z) = 0$.

For any positive integer n, define the completed tensor power $(U_\hbar\mathfrak{g})^{\hat{\otimes}n}$ as the quotient of $T(\mathfrak{g}^n)[[\hbar]]$ by the ideal generated by the usual n copies of the above relations.

The formulas

$$\Delta(K) = K \otimes 1 + 1 \otimes K, \tag{17}$$

$$\Delta(h^+[r]) = h^+[r] \otimes 1 + 1 \otimes h^+[r],$$
$$\Delta(h^-(z)) = h^-(z) \otimes 1 + 1 \otimes (q^{-K_1\partial}h^-)(z), \tag{18}$$

$$\Delta(e(z)) = e(z) \otimes q^{((T+U)h^+)(z)} + 1 \otimes e(z), \tag{19}$$

$$\Delta(f(z)) = f(z) \otimes 1 + q^{-h^-(z)} \otimes (q^{-K_1\partial} f)(z), \tag{20}$$

$$\Delta(D) = D \otimes 1 + 1 \otimes D + \sum_{i\in\mathbb{N}} \frac{\hbar}{4} h^+[e^i] \otimes h^+[Ae_i], \tag{21}$$

$r \in R$, for the coproduct,

$$\varepsilon(h^+[r]) = \varepsilon(h^-[\lambda]) = \varepsilon(x[\epsilon]) = \varepsilon(D) = \varepsilon(K) = 0, \tag{22}$$

$x = e, f$, $r \in R, \lambda \in \Lambda, \epsilon \in k$, for the counit,

$$\begin{aligned} &S(h^+[r]) = -h^+[r], \quad (Sh^-)(z) = -(q^{K\partial} h^-)(z), \\ &S(D) = -D + \frac{\hbar}{4} \sum_{i\in\mathbb{N}} h^+[e^i] h^+[Ae_i], \end{aligned} \tag{23}$$

$$\begin{aligned} (Se)(z) = -e(z) q^{((T+U)h^+)(z)}, \quad (Sf)(z) = -\left(q^{K\partial}(q^{h^-} f)\right)(z), \\ S(K) = -K, \end{aligned} \tag{24}$$

$r \in R$, for the antipode, define a topological (with respect to the completion introduced above) Hopf algebra structure on $U_\hbar\mathfrak{g}$.

Notation 2.1: We have posed $\delta(z,w) = \sum_{i\in\mathbb{Z}} \epsilon^i(z)\epsilon_i(w)$; this is an element of $k\bar{\otimes}k$. The indices R and Λ denote the projections on the first and second factor of the decomposition $k[[\hbar]] = R[[\hbar]] \oplus \Lambda[[\hbar]]$.

In (10), (11), we have attached indices z and w to the first and second factors of $k\bar{\otimes}k$. Recall that $a_0(z,w)(z_s - w_s)$ belongs to $\mathbb{C}((z_s, w_s))$.

K_1, K_2 respectively mean $K \otimes 1, 1 \otimes K$. The K, $f(z)$ and $h^-(z)$ used here correspond to $2K$, $\frac{1}{\hbar}(q^{-K\partial} f)(z)$ and $(q^{-K\partial} h^-)(z)$ of [8] respectively. ∎

Remark 2.4 (Variants of the vertex relations (10) and (11)): Due to the Hochschild cocycle properties explained in Remark 2.3, relations (10) and (11) are equivalent to the following ones,

$$\begin{aligned} &[\alpha(z) - \alpha(w) + \psi_-(z,w) a_0(z,w)(\alpha(z) - \alpha(w))] e(z) e(w) \\ &= q^{2(\tau-\phi)(z,w)} [\alpha(z) - \alpha(w) + \psi_+(z,w) a_0(z,w)(\alpha(z) - \alpha(w))] e(w) e(z), \forall \alpha \in k, \end{aligned} \tag{25}$$

$$\begin{aligned} &q^{2(\tau-\phi)(z,w)} [\alpha(z) - \alpha(w) + \psi_+(z,w) a_0(z,w)(\alpha(z) - \alpha(w))] f(z) f(w) \\ &= [\alpha(z) - \alpha(w) + \psi_-(z,w) a_0(z,w)(\alpha(z) - \alpha(w))] f(w) f(z), \ \forall \alpha \in k; \end{aligned} \tag{26}$$

note that for any $\alpha \in k$, $a_0(z,w)(\alpha(z) - \alpha(w))$ belongs to $\prod_{s,t\in S} \mathbb{C}((z_s, w_t)) = k\hat{\otimes}k$. ∎

Remark 2.5: Due to (3), the $e-e$ and $f-f$ relations (10) and (11) can be informally written as

$$
(27)\qquad \begin{aligned} e(z)e(w) &= q^{2\sum_i((T+U)e_i)(z)e^i(w)}e(w)e(z), f(z)f(w) \\ &= q^{2\sum_i e^i(z)((T+U)e_i)(w)}f(w)f(z). \qquad \blacksquare \end{aligned}
$$

Remark 2.6: The relations (14) can be rewritten as

$$
[D, q^{-((T+U)h^+)(z)}x(z)] = -\partial_z(q^{-((T+U)h^+)(z)}x(z)),
$$

$x=e,f$.

2.3 PRESENTATION OF $U_\hbar\bar{\mathfrak{g}}$. The Lie bialgebra $(\mathfrak{g},\bar{\delta})$ also admits a quantization. We denote by $U_\hbar\bar{\mathfrak{g}}$ the corresponding Hopf algebra. It is the quotient of $T(\mathfrak{g})[[\hbar]]$ by the following relations. Let us overline in the case of $U_\hbar\bar{\mathfrak{g}}$, the notation used in the case of $U_\hbar\mathfrak{g}$. The algebra relations are identical to those of $U_\hbar\mathfrak{g}$, except for those involving $\bar{h}^-[\lambda]$; we have

$$
(28)\qquad [\bar{h}^-[\lambda], \bar{e}(z)] = 2((T+U)\lambda)(z)\bar{e}(z),
$$

$$
(29)\qquad [\bar{h}^-[\lambda], \bar{f}(z)] = -2((T+U)(q^{-\bar{K}\partial}\lambda)_\Lambda)(z)\bar{f}(z),
$$

$$
(30)\qquad [\bar{h}^+[r], \bar{h}^-[\lambda]] = \frac{2}{\hbar}\langle (q^{\bar{K}\partial}-1)r, \lambda\rangle,
$$

$$
(31)\qquad \begin{aligned} &[\bar{h}^-[\lambda], \bar{h}^-[\lambda']] = \\ &\frac{2}{\hbar}\left(\langle U((q^{-\bar{K}\partial}\lambda)_\Lambda), q^{-\bar{K}\partial}\lambda'\rangle - \langle U\lambda, \lambda'\rangle - \langle T((q^{-\bar{K}\partial}\lambda)_R), q^{-\bar{K}\partial}\lambda'\rangle\ \right), \end{aligned}
$$

$$
(32)\qquad [\bar{e}(z), \bar{f}(w)] = \frac{1}{\hbar}[\delta(z,w)q^{((T+U)\bar{h}^+)(z)} - q^{-\bar{K}\partial_w}\left(q^{-\bar{h}^-(w)}\delta(z,w)\right),
$$

$$
(33)\qquad [\bar{D}, \bar{h}^-(z)] = -(\partial\bar{h}^-)(z) - [(1+q^{\bar{K}\partial})Ae_i](z)\bar{h}^+[e^i] + \bar{\mathcal{A}}(z),
$$

where

$$
(34)\qquad \begin{aligned} \bar{\mathcal{A}}(z) = &\sum_{i\in\mathbb{N}}[(1+q^{\bar{K}\partial})Ae_i](z)[(q^{\bar{K}\partial}-1)e_i](z) \\ &+\frac{2}{\hbar}[(1-q^{\bar{K}\partial})Ae_i](z)(q^{\bar{K}\partial}e^i)(z). \end{aligned}
$$

The coalgebra structure of $U_\hbar\bar{\mathfrak{g}}$ is defined by the coproduct

$$
(35)\qquad \bar{\Delta}(\bar{K}) = \bar{K}\otimes 1 + 1\otimes\bar{K},
$$

$$
(36)\qquad \begin{aligned} \bar{\Delta}(\bar{h}^+[r]) &= \bar{h}^+[r]\otimes 1 + 1\otimes\bar{h}^+[r], \\ \bar{\Delta}(\bar{h}^-(z)) &= (q^{\bar{K}_2\partial}\bar{h}^-)(z)\otimes 1 + 1\otimes\bar{h}^-(z), \end{aligned}
$$

$$\bar{\Delta}(\bar{e}(z)) = (q^{\bar{K}_2\partial}\bar{e})(z) \otimes q^{-\bar{h}^-(z)} + 1 \otimes \bar{e}(z), \tag{37}$$

$$\bar{\Delta}(\bar{f}(z)) = \bar{f}(z) \otimes 1 + q^{((T+U)\bar{h}^+)(z)} \otimes \bar{f}(z), \tag{38}$$

$$\bar{\Delta}(\bar{D}) = \bar{D} \otimes 1 + 1 \otimes \bar{D} + \sum_{i\in\mathbb{N}} \frac{\hbar}{4}\bar{h}^+[e^i] \otimes \bar{h}^+[Ae_i], \tag{39}$$

$r \in R$, the counit

$$\bar{\varepsilon}(\bar{h}^+[r]) = \bar{\varepsilon}(\bar{h}^-[\lambda]) = \bar{\varepsilon}(\bar{x}[\epsilon]) = \bar{\varepsilon}(\bar{D}) = \bar{\varepsilon}(\bar{K}) = 0, \tag{40}$$

$x = e, f$, $r \in R, \lambda \in \Lambda, \epsilon \in k$, and the antipode

$$\begin{gathered} \bar{S}(\bar{h}^+[r]) = -\bar{h}^+[r], \quad (\bar{S}\bar{h}^-)(z) = -(q^{-\bar{K}\partial}\bar{h}^-)(z), \\ \bar{S}(\bar{D}) = -\bar{D} + \frac{\hbar}{4}\sum_{i\in\mathbb{N}} \bar{h}^+[e^i]\bar{h}^+[Ae_i], \end{gathered} \tag{41}$$

$$\begin{gathered} (\bar{S}\bar{e})(z) = \left(q^{-\bar{K}\partial}(\bar{e}q^{\bar{h}^-})\right)(z), \quad (\bar{S}\bar{f})(z) = -q^{-((T+U)\bar{h}^+)(z)}\bar{f}(z), \\ \bar{S}(\bar{K}) = -\bar{K}, \end{gathered} \tag{42}$$

$r \in R$.

Then

Proposition 2.2 (see [8]): *Define $U_\hbar\mathfrak{g}_\pm$ and $U_\hbar\bar{\mathfrak{g}}_\pm$ as the subalgebras of $U_\hbar\mathfrak{g}$ and $U_\hbar\bar{\mathfrak{g}}$ generated by $\mathfrak{g}_\pm$ and $\bar{\mathfrak{g}}_\pm$, respectively. The pairs $(U_\hbar\mathfrak{g}_+, \Delta)$ and $(U_\hbar\mathfrak{g}_-, \Delta')$, as well as $(U_\hbar\bar{\mathfrak{g}}_+, \bar{\Delta})$ and $(U_\hbar\bar{\mathfrak{g}}_-, \bar{\Delta}')$, form dual Hopf algebras, quantizing the Lie bialgebra structures defined by $(\mathfrak{g}_\pm, \pm\delta)$ and $(\bar{\mathfrak{g}}_\pm, \pm\bar{\delta})$. The pairs $(U_\hbar\mathfrak{g}, \Delta)$ and $(U_\hbar\bar{\mathfrak{g}}, \Delta)$ are the double Hopf algebras of $(U_\hbar\mathfrak{g}_+, \Delta)$ and $(U_\hbar\bar{\mathfrak{g}}_+, \bar{\Delta})$ respectively, define quantizations of the Lie bialgebras $(\mathfrak{g}, \delta)$ and $(\bar{\mathfrak{g}}, \bar{\delta})$.*

Here Δ' denotes Δ composed with the permutation of factors.

Then

Proposition 2.3: *The map*

$$x[\epsilon] \mapsto \bar{x}[\epsilon], h^+[r] \mapsto \bar{h}^+[r], K \mapsto \bar{K}, D \mapsto \bar{D}, h^-[\lambda] \mapsto \bar{h}^-[(q^{K\partial}\lambda)_\Lambda],$$

$x = e, f$, $\epsilon \in k, r \in R, \lambda \in \Lambda$, *extends to an algebra isomorphism from $U_\hbar\mathfrak{g}$ to $U_\hbar\bar{\mathfrak{g}}$.*

In what follows, we will denote elements of $U_\hbar\bar{\mathfrak{g}}$ as elements of $U_\hbar\mathfrak{g}$, implicitly making use of this isomorphism.

3. PBW results for $U_\hbar\mathfrak{g}$

3.1 PBW RESULT FOR ALGEBRAS PRESENTED BY VERTEX RELATIONS. We will now prove a PBW statement, which was used implicitly in [8].

Let ζ be an indeterminate, and let V be the field of Laurent series $\mathbb{C}((\zeta))$. Let $\gamma_n = \zeta^n, n \in \mathbb{Z}$, and let us organize the $(\gamma_n)_{n\in\mathbb{Z}}$ in the generating series

$$\gamma(z) = \sum_{i\in\mathbb{Z}} \gamma_n z^{-n}.$$

Let $\hbar$ be a formal variable, and $\mathcal{A}$ be the quotient of $T(V)[[\hbar]]$ by the relations obtained as the Fourier modes of

$$(z - w + A(z,w))\gamma(z)\gamma(w) = (z - w + B(z,w))\gamma(w)\gamma(z), \tag{43}$$

for $A, B \in \hbar\mathbb{C}((z,w))[[\hbar]]$.

PROPOSITION 3.1: *Assume that A, B satisfy the relation*

$$(z - w + B(z,w))(w - z + B(w,z)) = (z - w + A(z,w))(w - z + A(w,z)); \tag{44}$$

then an element of $\mathcal{A}$ can be uniquely written as a sum

$$\sum_{p=0}^{k} \sum_{i_1<\cdots<i_p, \alpha_i\geq 1} \lambda_{i_1,\ldots,i_p}^{(\alpha_1,\ldots,\alpha_p)} \gamma_{i_1}^{\alpha_1} \cdots \gamma_{i_p}^{\alpha_p},$$

$k \geq 0$, $\lambda_{i_1,\ldots,i_p}^{(\alpha_1,\ldots,\alpha_p)}$ scalars, such that the number of indices

$$((i_1, \ldots, i_p), (\alpha_1, \ldots, \alpha_p))$$

with $i_1 = M$ and $\lambda_{i_1,\ldots,i_p}^{(\alpha_1,\ldots,\alpha_p)} \neq 0$, is finite for all M and zero for M large enough.

Proof: Let $\mathcal{A}_2$ be the span in $\mathcal{A}$ of infinite series $\sum_{n,m\geq N} a_{pq}e_p e_q$. (43) allows us to write, for any $n, m \in \mathbb{Z}$,

$$\rho_{n,m} = [\gamma_{n+1}, \gamma_m] - [\gamma_n, \gamma_{m+1}]$$

as a series in $\hbar\mathcal{A}_2[[\hbar]]$. We rearrange this system of relations in the following way. Let

$$\tau_{n-k,n+k} = \rho_{n-k,n+k} + \rho_{n-k+1,n+k-1} + \cdots + \rho_{n-1,n+1}$$

for $k > 0$,

$$\tau_{n+1-k,n+k} = \rho_{n+1-k,n+k} + \rho_{n+2-k,n+k-1} + \cdots + \rho_{n,n+1}$$

for $k \geq 1$,

$$\tau_{n+k,n-k} = \rho_{n+k,n-k} + \rho_{n+k-1,n-k+1} + \cdots + \rho_{n,n}$$

for $k \geq 0$,

$$\tau_{n+1+k,n-k} = \rho_{n+1+k,n-k} + \rho_{n+k,n-k+1} + \cdots + \rho_{n+1,n}$$

for $k \geq 0$; then the system of expressions for the $\rho_{n,m}$ is equivalent to as a system of expressions for the $\tau_{n,m}$.

Note that

$$\tau_{n-k,n+k} = -[\gamma_{n-k}, \gamma_{n+k+1}] + [\gamma_n, \gamma_{n+1}]$$

for $k > 0$,

$$\tau_{n+1-k,n+k} = [\gamma_{n+1-k}, \gamma_{n+k+1}]$$

for $k \geq 1$,

$$\tau_{n+k,n-k} = [\gamma_{n+k+1}, \gamma_{n-k}] - [\gamma_n, \gamma_{n+1}]$$

for $k \geq 0$,

$$\tau_{n+1+k,n-k} = [\gamma_{n+2+k}, \gamma_{n-k}]$$

for $k \geq 0$.

The expression for $\tau_{n,n}$ yields an expression for $[\gamma_n, \gamma_{n+1}]$. Substracting this expression to the expressions for $\tau_{n-k,n+k}$ and adding to the expression for $\tau_{n+k,n-k}$, we derive expressions for the $[\gamma_n, \gamma_m]$. This means that arbitrary monomials in the γ_i's expressed as linear combinations of the $(\gamma_{i_1}^{\alpha_1} \cdots \gamma_{i_p}^{\alpha_p})_{i_1 < \cdots < i_p, \alpha_i \geq 1}$ of the form described.

To prove that this expression is unique, we should check that the set of relations for $[\gamma_n, \gamma_m]$ is not redundant, i.e. that the expression for $[\gamma_n, \gamma_m]$ is actually the opposite to the one for $[\gamma_m, \gamma_n]$. For that is is necessary to make sure that the expressions deduced for the coefficients of $z^n w^m$ in

$$(z - w + A(z,w))\gamma(z)\gamma(w) = (z - w + B(z,w))\gamma(w)\gamma(z)$$

and

$$(w - z + A(w,z))\gamma(w)\gamma(z) = (w - z + B(w,z))\gamma(z)\gamma(w)$$

are equivalent. But these equations are actually equivalent, because from (44) follows the existence of $\kappa \in 1 + \hbar\mathbb{C}((z,w))[[\hbar]]$, such that

$$(z - w + A(z,w)) = -\kappa(z,w)(w - z + A(w,z))$$

and

$$(z - w + B(z,w)) = -\kappa(z,w)(w - z + B(w,z)).$$

The existence of κ in turn follows from the more general statement

LEMMA 3.1: *Let $A, B, C, D \in \hbar\mathbb{C}((z,w))[[\hbar]]$; expand them as $A = \sum_{i\geq 1} \hbar^i A^{(i)}$, $A^{(i)} \in \mathbb{C}((z,w))$, etc. Assume that $A^{(1)}(z,z) = C^{(1)}(z,z)$, $A^{(1)}(z,z) \neq B^{(1)}(z,z)$ and that the $A, \dots, D$ satisfy the identity*

$$(z-w+A)(z-w+B) = (z-w+C)(z-w+D),$$

then there exists $\kappa_0 \in 1 + \hbar\mathbb{C}((z,w))[[\hbar]]$ such that

$$z - w + A = \kappa_0(z-w+C), \quad z-w+D = \kappa_0(z-w+B). \tag{45}$$

Proof: We first prove the following structure result.

LEMMA 3.2: *Let $z-w+E$ belong to $z-w+\hbar\mathbb{C}((z,w))[[\hbar]]$, then there exist unique $e \in \hbar\mathbb{C}((w))[[\hbar]]$ and $\kappa_E \in 1 + \hbar\mathbb{C}((z,w))[[\hbar]]$, such that*

$$z - w + E = \kappa_E(z-w+e) \tag{46}$$

Proof: Let $E = \sum_{i\geq 1} \hbar^i E_i(z,w)$, $E_i(z,w) \in \mathbb{C}((z,w))$. Let us formally solve the equation

$$z - w + \sum_{i\geq 1} \hbar^i E_i(z,w) = 0$$

as follows. In the first order in $\hbar$, we get

$$z = w - \sum_{i\geq 1} \hbar^i E_i(z,w),$$

then

$$z = w - \sum_{i\geq 1} \hbar^i E_i(w - \sum_{i\geq 1} \hbar^i E_i(z,w), w),$$

etc. Let $\sigma_1(z,w)$, $\sigma_2(z,w)$, etc. be the sequence of formal series obtained in the right hand sides. This sequence converges in the $\hbar$-adic topology to a series depending on w only, $w - \sum_{i\geq 1} \hbar^i e_i(w)$. We also have

$$z - w + E = \kappa_k(z,w)(z - \sigma_k(z,w)),$$

with $\kappa_k \in 1 + \hbar\mathbb{C}((z,w))[[\hbar]]$. Since the $(\sigma_k)_{k\geq 1}$ converges in the $\hbar$-adic topology, so does $(\kappa_k)_{k\geq 1}$. Let κ_E be the limit of $(\kappa_k)_{k\geq 1}$, then κ_E satisfies (46). ∎

Let us now prove Lemma 3.1. Let

$$a, b, c, d \in \hbar\mathbb{C}((z,w))[[\hbar]], \quad \kappa_A, \kappa_B, \kappa_C, \kappa_D \in 1 + \hbar\mathbb{C}((z,w))[[\hbar]]$$

be associated with A, B, C, D by Lemma 3.2, then (3.1) is rewritten as

$$\kappa_A \kappa_B (z-w+a)(z-w+b) = \kappa_C \kappa_D (z-w+c)(z-w+d).$$

Replacing z by $w-c$, then by $w-d$ in the l.h.s. of this equation, we get the equations (in $\mathbb{C}((w))[[\hbar]]$)

$$(a-c)(b-c)=0, \quad (a-d)(b-d)=0.$$

Since a,b,c,d are of the form $\hbar A^{(1)}+o(\hbar), \hbar B^{(1)}+o(\hbar), \hbar C^{(1)}+o(\hbar), \hbar D^{(1)}+o(\hbar)$, we cannot have $a=d$ or $b=c$, so that $a=b$ and $b=d$, and $\kappa_A\kappa_B=\kappa_C\kappa_D$. We then obtain the conclusion of the lemma, with $\kappa_0=\kappa_C/\kappa_A$. ∎

3.2 PBW RESULT FOR $U_\hbar\mathfrak{g}$. Let $\mathcal{B}^+$ be the quotient of the algebra $T(k)[[\hbar]]$ by the following relations: let $e'[\epsilon]$ denote the element of $T(k)[[\hbar]]$ corresponding to $\epsilon\in k$, and $e'(z)=\sum_{i\in\mathbb{Z}}e'[\epsilon^i]\epsilon_i(z)$, the relations are the Fourier coefficients of (10), with the series replaced by their analogues with primes.

Similarly, let $\mathcal{B}^-$ be the quotient of the algebra $T(k)[[\hbar]]$ by the following relations: let $f'[\epsilon]$ denote the element of $T(k)[[\hbar]]$ corresponding to $\epsilon\in k$, and $f'(z)=\sum_{i\in\mathbb{Z}}f'[\epsilon^i]\epsilon_i(z)$, the relations are the Fourier coefficients of (11), with the series replaced by their analogues with primes.

Finally, let $\mathcal{B}^0$ be the quotient of the algebra $T(k\oplus\mathbb{C}K'\oplus\mathbb{C}D')[[\hbar]]$ by the following relations: let for $\epsilon\in k$, $h[\epsilon]$ denote the element of $T(k\oplus\mathbb{C}K'\oplus\mathbb{C}D')[[\hbar]]$ corresponding to ϵ, the relations are (5), (6), (13), with the $h^+[r], h^-[\lambda]$ replaced by $h'[r], h'[\lambda]$.

There are algebra morphisms from $\mathcal{B}^\pm$, $\mathcal{B}^0$ to $U_\hbar\mathfrak{g}$, associating to each generator its version without prime.

We then have

LEMMA 3.3: *The composition of the above algebra morphisms with the multiplication of $U_\hbar\mathfrak{g}$ defines a linear map*

$$i_\mathfrak{g}:\mathcal{B}^+\hat{\otimes}\mathcal{B}^0\hat{\otimes}\mathcal{B}^-\to U_\hbar\mathfrak{g},$$

which is a linear isomorphism (the tensor products are completed over $\mathbb{C}[[\hbar]]$).

Proof: We first consider the case of the algebra $U_\hbar\mathfrak{g}'$, defined as the algebra with the same generators (except D) and relations as $U_\hbar\mathfrak{g}$. Let $\mathcal{B}'^0$ be the analogue of algebra $\mathcal{B}^0$ without generator D'. In that case, we obtain easily that $(b_i^+b_j'^0b_k^-)_{i,j,k}$ is a base of $U_\hbar\mathfrak{g}'$, if $(b_i^+)_i, (b_j'^0)_j, (b_k^-)_k$ are images of bases of $\mathcal{B}^+, \mathcal{B}'^0, \bar{\mathcal{B}}^-$.

Then we check that the r.h.s. of formulas (13), (14) define a derivation of $U_\hbar\mathfrak{g}'$. We then apply the PBW result for crossed products of algebras by derivations, and obtain for $U_\hbar\mathfrak{g}$ a base $(b_i^+b_j'^0b_k^-D^s)_{i,j,k;s\geq 0}$. We finally make use of (14), $x^\pm=f$, to pass (by triangular transformations) from this base to $(b_i^+b_j'^0D^sb_k^-)_{i,j,k;s\geq 0}$.

Since $(b_j'^0 D^s)_{j;s\geq 0}$ is the image of a base of $\mathcal{B}^0$, this final base has the desired form. ∎

LEMMA 3.4: *$\mathcal{B}^\pm$ are topologically spanned by $e'[\epsilon_{i_1}]^{\alpha_1}\cdots e'[\epsilon_{i_p}]^{\alpha_p}$, $i_1 < \cdots < i_p$, $\alpha_i \geq 1$, resp. $f'[\epsilon_{i_1}]^{\alpha_1}\cdots f'[\epsilon_{i_p}]^{\alpha_p}$, $i_1 < \cdots < i_p$, $\alpha_i \geq 1$.*

Proof: This follows directly from the analogue of Proposition 3.1 (where $\mathbb{C}((\zeta))$ is replaced by a direct sum $\oplus_{s\in S}\mathbb{C}((\zeta_s)))$, the fact that $(z_s-w_s)a_0 \neq -(z_s-w_s)a_0$, and the computation of [8] preceding Theorem 5:

$$\begin{aligned}&e^{2\hbar[-\psi_0(\gamma,\partial_z\gamma,\ldots)+\tau]}[z-w+\beta\psi_-(\gamma,\partial_z\gamma,\ldots)]\cdot\\&[e^{2\hbar[-\psi_0(\gamma,\partial_z\gamma,\ldots)+\tau]}(z-w+\beta\psi_-(\gamma,\partial_z\gamma,\ldots))]^{\sim}\\&=[z-w+\beta\psi_+(\gamma,\partial_z\gamma,\ldots)]\cdot[z-w+\beta\psi_+(\gamma,\partial_z\gamma,\ldots)]^{\sim},\end{aligned}$$

where $\beta(z,w) = (z-w)a_0$, and this is in turn written

$$e^{2\hbar(-\psi_0-\widetilde{\psi}_0)}e^{2\hbar\frac{T_z-T_w}{\partial_z+\partial_w}(\gamma-\widetilde{\gamma})}\frac{1+G\psi_-(\gamma,\partial_z\gamma,\ldots)}{1+G\psi_+(\gamma,\partial_z\gamma,\ldots)}\frac{1-G\psi_-(\widetilde{\gamma},\partial_w\widetilde{\gamma},\ldots)}{1-G\psi_+(\widetilde{\gamma},\partial_w\widetilde{\gamma},\ldots)}=1,$$

which amounts to the statement (3.17) of [8]. ∎

PROPOSITION 3.2: *The*

$$e[\epsilon_{i_1}]^{\alpha_1}\cdots e[\epsilon_{i_p}]^{\alpha_p}h[\epsilon_{k_1}]^{\gamma_1}\cdots h[\epsilon_{k_r}]^{\gamma_r}D^dK^tf[\epsilon_{j_1}]^{\beta_1}\cdots f[\epsilon_{j_q}]^{\beta_q}$$

with $i_1 < \cdots < i_p$, $\alpha_i \geq 1$, $j_1 < \cdots < j_q$, $\beta_i \geq 1$, $k_1 < \cdots < k_r$, $\gamma_i \geq 1$, $d,t \geq 0$, form a topological basis of $U_\hbar\mathfrak{g}$.

Proof: This follows from Lemmas 3.3, 3.4 and the fact that

$$h'[\epsilon_{k_1}]^{\gamma_1}\cdots h'[\epsilon_{k_r}]^{\gamma_r}D^dK^t,\quad k_1<\cdots<k_r,\quad \gamma_i\geq 1,\quad d,t\geq 0,$$

forms a basis of $\mathcal{B}^0$. ∎

Remark 3.1: It is straightforward to repeat the resoning above to obtain topological bases of the $U_\hbar\mathfrak{g}^{\hat{\otimes}N}$, as tensor powers of the base obtained in Proposition 3.2.

Remark 3.2: It would be interesting to explicitly compute, in the case of the algebras $\mathcal{B}^\pm$, the κ_0 provided by Proposition 3.1.

4. Subalgebra $U_\hbar\mathfrak{g}_R$ of $U_\hbar\mathfrak{g}$

4.1 PRESENTATION OF $U_\hbar\mathfrak{g}_R$. Recall that $\mathfrak{g}$ contains as a Lie subalgebra $\mathfrak{g}_R$. In this section, we define a subalgebra $U_\hbar\mathfrak{g}_R$ of $U_\hbar\mathfrak{g}$, such that the inclusion $U_\hbar\mathfrak{g}_R \subset U_\hbar\mathfrak{g}$ is a deformation of $U\mathfrak{g}_R \subset U\mathfrak{g}$.

Let $U_\hbar\mathfrak{g}_R$ be the algebra with generators $\widetilde{D}, \widetilde{e}[r], \widetilde{f}[r], \widetilde{h}[r], R, r \in R$ and relations

$$\widetilde{x}[\alpha_1 r_1 + \alpha_2 r_2] = \alpha_1\widetilde{x}[r_1] + \alpha_2\widetilde{x}[r_2], \quad x = e, f, h, \tag{47}$$

$$[\widetilde{h}[r], \widetilde{h}[r']] = 0, \quad \forall r, r' \in R, \tag{48}$$

$$[\widetilde{h}[r], \widetilde{e}[r']] = 2\widetilde{e}[rr'], \quad [\widetilde{h}[r], \widetilde{f}[r']] = -2\widetilde{f}[rr'], \quad \forall r, r' \in R, \tag{49}$$

$$\begin{aligned}\widetilde{e}[r_1\alpha]\widetilde{e}[r_2] - \widetilde{e}[r_1]\widetilde{e}[\alpha r_2] + \widetilde{e}[r_1\psi_-^{(1)}\gamma(\alpha)^{(1)}]\widetilde{e}[r_2\psi_-^{(2)}\gamma(\alpha)^{(2)}] = \\ \widetilde{e}[r_2(q^{2(\tau-\phi)})^{(2)}]\widetilde{e}[\alpha r_1(q^{2(\tau-\phi)})^{(1)}] - \widetilde{e}[r_2(q^{2(\tau-\phi)})^{(2)}\alpha]\widetilde{e}[r_1(q^{2(\tau-\phi)})^{(1)}] \\ + \widetilde{e}[r_2(\psi_+q^{2(\tau-\phi)})^{(2)}\gamma(\alpha)^{(2)}]\widetilde{e}[r_1(\psi_+q^{2(\tau-\phi)})^{(1)}\gamma(\alpha)^{(1)}],\end{aligned} \tag{50}$$

$$\begin{aligned}\widetilde{f}[r_2]\widetilde{f}[\alpha r_1] - \widetilde{f}[r_2\alpha]\widetilde{f}[r_1] + \widetilde{f}[r_2(\psi_-)^{(2)}\gamma(\alpha)^{(2)}]\widetilde{f}[r_1(\psi_-)^{(1)}\gamma(\alpha)^{(1)}] = \\ \widetilde{f}[(q^{2(\tau-\phi)})^{(1)}r_1\alpha]\widetilde{f}[(q^{2(\tau-\phi)})^{(2)}r_2] - \widetilde{f}[(q^{2(\tau-\phi)})^{(1)}r_1]\widetilde{f}[(q^{2(\tau-\phi)})^{(2)}\alpha r_2] \\ + \widetilde{f}[r_1(q^{2(\tau-\phi)}\psi_+)^{(1)}\gamma(\alpha)^{(1)}]\widetilde{f}[r_2(q^{2(\tau-\phi)}\psi_+)^{(2)}\gamma(\alpha)^{(2)}],\end{aligned} \tag{51}$$

$$[\widetilde{e}[r_1], \widetilde{f}[r_2]] = \sum_{s\in S} \mathrm{res}_s\Big\{\frac{1}{\hbar}(r_1r_2)(z)q^{((T+U)\widetilde{h})(z)}\omega_z\Big\}, \tag{52}$$

$$[\widetilde{D}, \widetilde{h}[r]] = \widetilde{h}[\partial r], \tag{53}$$

$$[\widetilde{D}, \widetilde{x}^{\pm}[r]] = \widetilde{x}^{\pm}[\partial r] + \frac{\hbar}{2}\sum_{i\in\mathbb{N}} \widetilde{h}[e^i]\widetilde{x}^{\pm}[(Ae_i)r] \tag{54}$$

for $x = e, f, h$; $x^{\pm} = e, f$; $\alpha, r_i \in R$, α_i scalars, $i = 1, 2$; $\widetilde{h}(z) = \sum_{i\in\mathbb{N}}\widetilde{h}[e^i]e_i(z)$;

$$\gamma(\alpha) = (\alpha\otimes 1 - 1\otimes\alpha)a_0 \in R\otimes R$$

for $\alpha \in R$.

Notation 4.1: For $\xi \in R\otimes R$, $\xi = \sum_i \xi_i \otimes \xi_i'$, and $a, b \in R$, we denote $\sum_i \widetilde{x}[a\xi_i]\widetilde{x}[b\xi_i']$ as $\widetilde{x}[a\xi^{(1)}]\widetilde{x}[b\xi^{(2)}]$. The operator A arising in (54) has been defined in (15). Note that the sums arising in (54) have only finite non-zero terms, since U has the property that, for any sequence (ξ_i) with $\xi_i \to 0$, $U\xi_i$ is zero for i large enough, and both sequences $e_i, \partial e_i$ tend to zero; and, on the other hand, $(\partial e_i)_R = 0$ for i large enough. ∎

Remark 4.1 (On relations (50) and (51)): The complicated-looking formulas (50) and (51) are simply obtained by pairing the vertex relations (25) and (26), for $\alpha \in R$, with an element of $R \otimes R$.

Remark 4.2 (Generating series for relations (50), (51) and (52)): Let us introduce the generating series

$$\widetilde{e}(z) = \sum_{i \in \mathbb{N}} \widetilde{e}[e^i] e_i(z), \quad \widetilde{f}(z) = \sum_{i \in \mathbb{N}} \widetilde{f}[e^i] e_i(z).$$

Relations (50) and (51) can then be obtained as Fourier modes of

$$\begin{aligned}(\ [\alpha(z) - \alpha(w) + \psi_-(z,w)\gamma(\alpha)(z,w)]\widetilde{e}(z)\widetilde{e}(w) \ )_{\Lambda,\Lambda} \\ = \left(q^{2(\tau-\phi)(z,w)} [\alpha(z) - \alpha(w) + \psi_+(z,w)\gamma(\alpha)(z,w)]\widetilde{e}(w)\widetilde{e}(z) \right)_{\Lambda,\Lambda}, \quad \forall \alpha \in R,\end{aligned} \tag{55}$$

and

$$\begin{aligned}(\ q^{2(\tau-\phi)(z,w)} [\alpha(z) - \alpha(w) + \psi_+(z,w)\gamma(\alpha)(z,w)]\widetilde{f}(z)\widetilde{f}(w) \ )_{\Lambda,\Lambda} \\ = \left([\alpha(z) - \alpha(w) + \psi_-(z,w)\gamma(\alpha)(z,w)]\widetilde{f}(w)\widetilde{f}(z) \right)_{\Lambda,\Lambda}, \quad \forall \alpha \in R,\end{aligned} \tag{56}$$

where the index Λ, Λ has the following meaning: for any vector space V and $\xi \in V \otimes (k \bar{\otimes} k)$ (with $k \bar{\otimes} k = \lim_N k/k_N \otimes k/k_N$), $\xi_{\Lambda,\Lambda} = (id_V \otimes pr_\Lambda \otimes pr_\Lambda)\xi$, where pr_Λ denote the projection on the second summand of $k = R \oplus \Lambda$. In terms of these generating series, the relations (52) take the form

$$[\widetilde{e}(z), \widetilde{f}(w)] = \left(\frac{1}{\hbar} q^{((T+U)\widetilde{\hbar})(z)} \delta(z,w) \right)_{\Lambda,\Lambda}, \tag{57}$$

which can be rewritten as

$$\begin{aligned}[\widetilde{e}(z), \widetilde{f}(w)] &= \sum_{i \in \mathbb{N}} \left(\frac{1}{\hbar} q^{((T+U)\widetilde{\hbar})(z)} e^i(z) \right)_\Lambda e_i(w) \\ &= \sum_{i \in \mathbb{N}} \left(\frac{1}{\hbar} q^{((T+U)\widetilde{\hbar})(w)} e^i(w) \right)_\Lambda e_i(z)\end{aligned}$$

or in "mixed" form

$$[\widetilde{e}(z), \widetilde{f}[r]] = [\widetilde{e}[r], \widetilde{f}(z)] = \left(\frac{1}{\hbar} q^{((T+U)\widetilde{\hbar})(z)} r(z) \right)_\Lambda,$$

for any $r \in R$.

4.2 PBW RESULT FOR $U_\hbar\mathfrak{g}_R$ AND INCLUSION IN $U_\hbar\mathfrak{g}$. Let $\mathcal{B}_R^+$ be the algebra with generators $e''[r], r \in R$, and relations (47), with $\widetilde{x}$ replaced by e'', and (50), with $\widetilde{e}$ replaced by e''; let $\mathcal{B}_R^-$ be the algebra with generators $f''[r], r \in R$, and relations (47), with $\widetilde{x}$ replaced by f'', and (51), with $\widetilde{f}$ replaced by f''; and let $\mathcal{B}_R^0$ be the algebra with generators $D'', h''[r], r \in R$, and relations (47), with $\widetilde{x}$ replaced by h'', (48) and (53), with $\widetilde{D}, \widetilde{h}$ replaced by D'', h''.

LEMMA 4.1:

(1) *There are injective algebra morphisms $i^\pm, i^0$ from $\mathcal{B}_R^\pm, \mathcal{B}_R^0$ to $\mathcal{B}^\pm, \mathcal{B}^0$, sending each $x''[r]$ to $x'[r]$, and D'' to D', $x = e, f, h, r \in R$.*

(2) *Topological bases of $\mathcal{B}_R^\pm, \mathcal{B}_R^0$ are given by the*

$$(e''[e^{i_1}]^{\alpha_1} \cdots e''[e^{i_p}]^{\alpha_p})_{i_1 < \cdots < i_p, \alpha_i \geq 1}, \quad (f''[e^{i_1}]^{\alpha_1} \cdots f''[e^{i_p}]^{\alpha_p})_{i_1 < \cdots < i_p, \alpha_i \geq 1},$$

and $(h''[e^{i_1}]^{\alpha_1} \cdots h''[e^{i_p}]^{\alpha_p} D^s)_{i_1 < \cdots < i_p, \alpha_i \geq 1, s \geq 0}$.

Proof: (1) (25) [resp. (26)] with e (resp. f) replaced by e' (resp. f'), are relations of $\mathcal{B}^+$ (resp. of $\mathcal{B}^-$). Pair them with $r_1(z)r_2(w)$. We obtain relations (50), (51), with e', f' instead of $\widetilde{e}, \widetilde{f}$. This shows that the maps $x''[r] \mapsto x'[r]$, $x = e, f$ extend to morphisms from $\mathcal{B}_R^\pm$ to $\mathcal{B}^\pm$. The statement on the map $h''[r] \mapsto h'[r], D'' \mapsto D'$ is evident.

(2) The case of $\mathcal{B}_R^0$ is obvious. Let us treat $\mathcal{B}_R^+$. From relations (50) follows

$$[e''[r_0\alpha], e''[r_1]] - [e''[r_0], e''[\alpha r_1]] \in \hbar\mathcal{B}_R^+, \tag{58}$$

for any $r_0, r_1, \alpha \in R$. We then get, setting $r_0 = 1$ in (58),

$$[e''[\alpha], e''[\beta]] \in [e''[1], e''[\alpha\beta]] + \hbar\mathcal{B}_R^+;$$

on the other hand, setting $r_0 = r_1 = 1$ in (58), we find that

$$2[e''[\alpha], e''[1]] \in \hbar\mathcal{B}_R^+$$

so that any commutator $[e''[\alpha], e''[\beta]]$ is in $\hbar\mathcal{B}_R^+$. This shows that any monomial can be transformed into a combination of the $e''[e_{i_1}]^{\alpha_1} \cdots e''[e_{i_p}]^{\alpha_p}$, with $i_1 < \cdots < i_p$, and $\alpha_i \geq 1$.

Suppose now that some combination

$$\sum_{i \geq 0} \hbar^i \sum_{i_1 < \cdots < i_p, \alpha_i \geq 1} a^{(i)}_{i_j, \alpha_j} e''[e^{i_1}]^{\alpha_1} \cdots e''[e^{i_p}]^{\alpha_p}$$

is zero (where for each i, the second sum is a finite one). Applying i^+ to this identity, we obtain the identity in $\mathcal{B}^+$

$$\sum_{i \geq 0} \hbar^i \sum_{i_1 < \cdots < i_p, \alpha_i \geq 1} a^{(i)}_{i_j, \alpha_j} e'[e^{i_1}]^{\alpha_1} \cdots e'[e^{i_p}]^{\alpha_p} = 0,$$

which implies that all $a^{(i)}_{i_j,\alpha_j}$ are zero due to Lemma 3.4. This shows that

$$(e''[e^{i_1}]^{\alpha_1}\cdots e''[e^{i_p}]^{\alpha_p})_{i_1<\cdots<i_p,\alpha_i\geq 1}$$

is topologically free in $\mathcal{B}^+_R$. Part 2 of the lemma follows for $\mathcal{B}^+_R$. The case of $\mathcal{B}^-_R$ is similar. ∎

LEMMA 4.2: *There are injective algebra morphisms from $\mathcal{B}^\pm_R$ and $\mathcal{B}^0_R$ to $U_\hbar\mathfrak{g}_R$, sending each $x''[r]$ to $\widetilde{x}[r]$, and D'' to $\widetilde{D}$, $x=e,f,h,r\in R$. The composition of the tensor product of these morphisms, with the multiplication of $U_\hbar\mathfrak{g}_R$, induces a linear isomorphism*

$$i_{\mathfrak{g}_R}:\mathcal{B}^+_R\hat{\otimes}\mathcal{B}^0_R\hat{\otimes}\mathcal{B}^-_R\to U_\hbar\mathfrak{g}_R.$$

Proof: Let $U_\hbar\mathfrak{g}'_R$ be the algebra with the same generators (without $\widetilde{D}$) and relations as $U_\hbar\mathfrak{g}_R$. We can prove by direct computation that the r.h.s. of relations (53), (54) define derivations of this algebra.

The proof of the lemma is then identical to that of Lemma 3.3. ∎

PROPOSITION 4.1: *The map sending each $\widetilde{x}[r]$ to $x[r]$, $x=e,f,h$, $r\in R$, and $\widetilde{D}$ to D, extends to an injective algebra morphism from $U_\hbar\mathfrak{g}_R$ to $U_\hbar\mathfrak{g}$.*

Proof: Let us first show that this map extends to an algebra morphism for $U_\hbar\mathfrak{g}_R$ to $U_\hbar\mathfrak{g}$. For any $\alpha\in R$, (25) and (26) are relations of $U_\hbar\mathfrak{g}$. Pairing them with $r_1(z)r_2(w)$ $(r_1,r_2\in R)$, we obtain relations (50), (51), with $\widetilde{e},\widetilde{f}$ replaced by e,f. Moreover, (6) is a relation of $U_\hbar\mathfrak{g}$; pairing it with $r_1(z)r_2(w)$, we obtain (48) with $\widetilde{x}$ replaced by x for $x=e,f,h$. Finally, pairing relations (14), (13) with $r(z)$, $r\in R$, we obtain relations (54), (53), with $\widetilde{x}$ replaced by x. This shows that the map $\widetilde{x}[r]\mapsto x[r],\widetilde{D}\mapsto D$ extends to an algebra morphism from $U_\hbar\mathfrak{g}_R$ to $U_\hbar\mathfrak{g}$, which we will denote by ι.

We easily check that the diagram

$$\begin{array}{ccc}\mathcal{B}^+_R\otimes\mathcal{B}^0_R\otimes\mathcal{B}^-_R & \xrightarrow{i_{\mathfrak{g}_R}} & U_\hbar\mathfrak{g}_R\\ \Big\downarrow i^+\otimes i^0\otimes i^- & & \Big\downarrow\iota\\ \mathcal{B}^+\otimes\mathcal{B}^0\otimes\mathcal{B}^- & \xrightarrow{i_{\mathfrak{g}}} & U_\hbar\mathfrak{g}\end{array}\tag{59}$$

commutes. By Lemmas 3.3 and 4.2, the horizontal arrows are vector spaces isomorphisms. From Lemma 4.1 follows that the left vertical arrow is injective. It follows that ι is also injective. ∎

4.3 DEPENDENCE OF $U_\hbar\mathfrak{g}_R$ IN τ AND Λ. In [8] we showed that the various algebras $U_\hbar\mathfrak{g}$, associated to different choices of Λ and τ, are all isomorphic. We are now going to show that the same is true for their subalgebras $U_\hbar\mathfrak{g}_R$. We will denote by a superscript (Λ,τ) the objects associated with a choice (Λ,τ).

We will study two families of changes of the pair (Λ,τ) that will generate all possible changes. The first is to change (Λ,τ) into (Λ,τ'), where $v=\tau'-\tau$ is an arbitrary antisymmetric element in $R^{\otimes 2}[[\hbar]]$. To it is associated the map $u:\Lambda[[\hbar]]\to R[[\hbar]]$ defined by $u(\lambda)=\langle v,1\otimes\lambda\rangle_k$. The second family of changes is parametrized by some continuous anti-self-adjoint linear map $r:\Lambda\to R$ (or equivalently, by some antisymmetric r_0 in $R^{\otimes 2}$). We then define a new pair $(\bar\Lambda,\bar\tau)$ by the formulas $\bar\Lambda=(1+r)\Lambda$ and $\bar\tau=\tau-\sum_{i\in\mathbb{N}}T(r(e_i))\otimes e^i$.

Recall now the results of [8].

PROPOSITION 4.2 (see [8]):

(1) *There is an isomorphism $i^{\tau,\tau'}$ from $U_\hbar\mathfrak{g}^{(\Lambda,\tau')}$ to $U_\hbar\mathfrak{g}^{(\Lambda,\tau)}$, such that*

$$i^{\tau,\tau'}(e^{(\Lambda,\tau')}(z))=e^{\frac{\hbar}{2}(uh^{+(\Lambda,\tau)})(z)}e^{(\Lambda,\tau)}(z),$$
$$i^{\tau,\tau'}(e^{(\Lambda,\tau')}(z))=f^{(\Lambda,\tau)}(z)e^{\frac{\hbar}{2}(uh^{+(\Lambda,\tau)})(z)},$$
$$i^{\tau,\tau'}(h^{+(\Lambda,\tau')}(z))=h^{+(\Lambda,\tau)}(z),\quad i^{\tau,\tau'}(D^{(\Lambda,\tau')})=D^{(\Lambda,\tau)}.$$

(2) *There is an isomorphism $i^{\Lambda,\bar\Lambda}$ from $U_\hbar\mathfrak{g}^{(\Lambda,\tau)}$ to $U_\hbar\mathfrak{g}^{(\bar\Lambda,\bar\tau)}$ such that*

$$i^{\Lambda,\bar\Lambda}(x^{(\Lambda,\tau)}(z))=x^{(\bar\Lambda,\bar\tau)}(z),\quad i^{\Lambda,\bar\Lambda}(D^{(\Lambda,\tau)})=D^{(\bar\Lambda,\bar\tau)},$$

$x=e,f,h^+$.

PROPOSITION 4.3: *Both maps $i^{\tau,\tau'}$ and $i^{\Lambda,\bar\Lambda}$ restrict to isomorphisms of $U_\hbar\mathfrak{g}_R^{(\Lambda,\tau)}$ with $U_\hbar\mathfrak{g}_R^{(\Lambda,\tau')}$ and $U_\hbar\mathfrak{g}_R^{(\bar\Lambda,\bar\tau)}$. Therefore the algebras $U_\hbar\mathfrak{g}_R^{(\Lambda,\tau)}$ are isomorphic for all choices of (Λ,τ).*

Proof: For $r\in R$

$$i^{\tau,\tau'}(e^{(\Lambda,\tau')}[r])=\sum_{s\in S}\mathrm{res}_s\left(re^{\frac{\hbar}{2}(uh^{+(\Lambda,\tau)})}e^{(\Lambda,\tau)}\right)\omega$$
$$=\sum_{s\in S}\mathrm{res}_s\sum_{n\geq 0}\frac{1}{n!}\sum_{i_1,\dots,i_n\in\mathbb{N}}\left(\frac{\hbar}{2}\right)^n h^{+(\Lambda,\tau)}[e^{i_1}]\cdots h^{+(\Lambda,\tau)}[e^{i_n}](ru(e_{i_1})\cdots u(e_{i_n})e^{(\Lambda,\tau)})(z)\omega_z$$
$$=\sum_{s\in S}\mathrm{res}_s\sum_{n\geq 0}\frac{1}{n!}\sum_{i_1,\dots,i_n\in\mathbb{N}}\left(\frac{\hbar}{2}\right)^n h^{+(\Lambda,\tau)}[e^{i_1}]\cdots h^{+(\Lambda,\tau)}[e^{i_n}]e^{(\Lambda,\tau)}[r\cdot u(e_{i_1})\cdots u(e_{i_n})]$$

so $i^{\tau,\tau'}(e^{(\Lambda,\tau')}[r]) \in U_\hbar\mathfrak{g}_R^{(\Lambda,\tau)}$. The proof that $i^{\tau,\tau'}(f^{(\Lambda,\tau')}[r]) \in U_\hbar\mathfrak{g}_R^{(\Lambda,\tau)}$ is similar and in the case of $i^{\tau,\tau'}(h^{+(\Lambda,\tau')}[r])$ the analogous statement is obvious. The inverse of $i^{\tau,\tau'}$ is $i^{\tau',\tau}$; in particular, $i^{\tau,\tau'}$ is bijective. It is also an algebra morphism by Proposition 4.2. This shows (1). The proof of (2) is obvious. ∎

Remark 4.3: There is another isomorphism $\bar{i}^{\tau,\tau'}$ from $U_\hbar\mathfrak{g}^{(\Lambda,\tau)}$ to $U_\hbar\mathfrak{g}^{(\Lambda,\tau')}$, such that

$$\bar{i}^{\tau,\tau'}(e^{(\Lambda,\tau')}(z)) = e^{\frac{\hbar}{2}(uh^+)(z)}e^{(\Lambda,\tau)}(z), \quad \bar{i}^{\tau,\tau'}(e^{(\Lambda,\tau')}(z)) = \bar{f}^{(\Lambda,\tau)}(z)e^{\frac{\hbar}{2}(uh^+)(z)},$$
$$\bar{i}^{\tau,\tau'}(h^{+(\Lambda,\tau')}(z)) = h^{+(\Lambda,\tau)}(z), \quad \bar{i}^{\tau,\tau'}(D^{(\Lambda,\tau')}) = D^{(\Lambda,\tau)}.$$

It is easy to see that it also yields an isomorphism from $U_\hbar\mathfrak{g}^{(\Lambda,\tau)}$ to $U_\hbar\mathfrak{g}^{(\Lambda,\tau')}$.

4.4 $U_\hbar\mathfrak{g}_R$ AND $\Delta, \bar{\Delta}$. Let us define $U_\hbar\mathfrak{g}_R\hat{\otimes}U_\hbar\mathfrak{g}$, resp. $U_\hbar\mathfrak{g}\hat{\otimes}U_\hbar\mathfrak{g}_R$, as the quotients of $T(\mathfrak{g}_R\oplus\mathfrak{g})[[\hbar]]$, resp. $T(\mathfrak{g}\oplus\mathfrak{g}_R)[[\hbar]]$ by the usual relations. These are complete subalgebras of $U_\hbar\mathfrak{g}\hat{\otimes}U_\hbar\mathfrak{g}$.

PROPOSITION 4.4:

$$\Delta(U_\hbar\mathfrak{g}_R) \subset U_\hbar\mathfrak{g}\hat{\otimes}U_\hbar\mathfrak{g}_R, \quad \bar{\Delta}(U_\hbar\mathfrak{g}_R) \subset U_\hbar\mathfrak{g}_R\hat{\otimes}U_\hbar\mathfrak{g}.$$

Proof: It is enough to check these statements for the generators of $U_\hbar\mathfrak{g}_R$. They are obvious for D and $h^+[r]$. Moreover,

$$\Delta(e[r]) = \sum_{s\in S}\operatorname{res}_s\left(e(z)\otimes q^{((T+U)h^+)(z)})r(z)\omega_z\right) + 1\otimes e[r].$$

The first term of the r.h.s. of this equality can be decomposed as a sum of terms, the second factors of which all lie in the algebra generated by the $h^+[r'], r' \in R$.

We also have

$$\begin{aligned}\Delta(f[r]) &= f[r]\otimes 1 + \sum_{s\in S}\operatorname{res}_s\left(q^{-h^-(z)}\otimes(q^{-K_1\partial}f)(z)\right)r(z)\omega_z \\ &= f[r]\otimes 1 + \sum_{s\in S}\operatorname{res}_s\Big(\sum_{p\geq 0}\sum_{i_1,\ldots,i_p\in\mathbb{N}}\frac{(-\hbar)^p}{p!}h^-[e_{i_1}]\cdots h^-[e_{i_p}]e^{i_1}(z) \\ &\qquad \cdots e^{i_p}(z) \\ &\qquad \otimes(q^{-K_1\partial}f)(z)\Big)r(z)\omega_z \\ &= f[r]\otimes 1 + \sum_{p\geq 0}\sum_{i_1,\ldots,i_p\in\mathbb{N}}\frac{(-\hbar)^p}{p!}h^-[e_{i_1}]\cdots h^-[e_{i_p}]\otimes f[q^{K_1\partial}(e^{i_1}\cdots e^{i_p}r)].\end{aligned}$$

All $f[\partial^s(e^{i_1}\cdots e^{i_p}r)]$ belong to $U_\hbar\mathfrak{g}_R$. This ends the proof of the proposition in the case of Δ.

In the case of $\bar\Delta$, the proof is similar. ∎

5. F, universal R-matrices and Hopf algebra pairings

In what follows, we will denote by $U_\hbar\mathfrak{n}_\pm$ the algebras $\mathcal{B}^\pm$.

Recall that the Hopf algebras $(U_\hbar\mathfrak{g}_+,\Delta)$ and $(U_\hbar\mathfrak{g}_-,\Delta')$, as well as $(U_\hbar\bar{\mathfrak{g}}_+,\bar\Delta)$ and $(U_\hbar\bar{\mathfrak{g}}_-,\bar\Delta')$, are dual. Denote by $\langle,\rangle_{U_\hbar\mathfrak{g}}$ and $\langle,\rangle_{U_\hbar\bar{\mathfrak{g}}}$ the corresponding bilinear forms. They are defined by the formulas

$$(60)\qquad \langle h^+[r],h^-[\lambda]\rangle_{U_\hbar\mathfrak{g}}=\frac{2}{\hbar}\langle r,\lambda\rangle_k,\quad \langle e[\epsilon],f[\eta]\rangle_{U_\hbar\mathfrak{g}}=\frac{1}{\hbar}\langle\epsilon,\eta\rangle_k,$$

for $\epsilon,\eta\in k$, $r\in R,\lambda\in\Lambda$,

$$\langle D,K\rangle_{U_\hbar\mathfrak{g}}=1,\quad \langle D,\mathfrak{a}(k)\rangle_{U_\hbar\mathfrak{g}}=\langle\mathfrak{a}(k),K\rangle_{U_\hbar\mathfrak{g}}=0,$$

and

$$(61)\qquad \langle h^+[r],h^-[\lambda]\rangle_{U_\hbar\bar{\mathfrak{g}}}=\frac{2}{\hbar}\langle r,\lambda\rangle_k,\quad \langle f[\epsilon],e[\eta]\rangle_{U_\hbar\bar{\mathfrak{g}}}=\frac{1}{\hbar}\langle\epsilon,\eta\rangle_k,$$

for $\epsilon,\eta\in k$, $r\in R,\lambda\in\Lambda$,

$$\langle D,K\rangle_{U_\hbar\bar{\mathfrak{g}}}=1,\quad \langle D,\mathfrak{a}(k)\rangle_{U_\hbar\bar{\mathfrak{g}}}=\langle\mathfrak{a}(k),K\rangle_{U_\hbar\bar{\mathfrak{g}}}=0.$$

From the Hopf algebra pairing rules follows immediately

LEMMA 5.1: *Let $U_\hbar\mathfrak{h}_R$ be the subalgebra of $U_\hbar\mathfrak{g}$ generated by D and the $h^+[r],r$ in R, and let $U_\hbar\mathfrak{h}_\Lambda$ be the subalgebra of $U_\hbar\mathfrak{g}$ generated by K and the $h^+[\lambda],\lambda$ in Λ.*

For any $x^\pm$ in $U_\hbar\mathfrak{n}_\pm$ and t_R,t_Λ in $U_\hbar\mathfrak{h}_R$ and $U_\hbar\mathfrak{h}_\Lambda$, we have

$$(62)\qquad \langle t_Rx^+,t_\Lambda x^-\rangle_{U_\hbar\mathfrak{g}}=\varepsilon(t_R)\varepsilon(t_\Lambda)\langle x^+,x^-\rangle_{U_\hbar\mathfrak{g}}$$

and

$$(63)\qquad \langle x^-t_R,x^+t_\Lambda\rangle_{U_\hbar\bar{\mathfrak{g}}}=\varepsilon(t_R)\varepsilon(t_\Lambda)\langle x^-,x^+\rangle_{U_\hbar\bar{\mathfrak{g}}}.$$

Then:

PROPOSITION 5.1: *The restrictions to $U_\hbar\mathfrak{n}_+\times U_\hbar\mathfrak{n}_-$and $U_\hbar\mathfrak{n}_-\times U_\hbar\mathfrak{n}_+$ of the pairings $\langle,\rangle_{U_\hbar\mathfrak{g}}$ and $\langle,\rangle_{U_\hbar\bar{\mathfrak{g}}}$ coincide up to permutation.*

Proof: Fix ϵ_i, η_j in k, $i = 1, \dots, n$, $j = 1, \cdots, m$. Let us compute

$$\langle \prod_{i=1}^{n} e[\epsilon_i], \prod_{j=1}^{m} f[\eta_j] \rangle_{U_\hbar \mathfrak{g}}$$

(we denote by $\prod_{i \in I} x_i$ the product $x_{i_1} \cdots x_{i_p}$, where I is a set of integers $\{i_\alpha\}$ with $i_1 < i_2 < \cdots$). It is clear that this is zero if n is not equal to m. Assume that $n = m$, then let us compute the generating series $\langle \prod_{i=1}^{n} e[\epsilon_i], \prod_{j=1}^{m} f(z_j) \rangle_{U_\hbar \mathfrak{g}}$. By the Hopf algebra pairing rules, it is equal to

$$\sum_{\sigma \in S_n} \Big\langle \bigotimes_{i=1}^{n} e[\epsilon_i], \bigotimes_{i=1}^{n} \Big\{ \prod_{l=1, \sigma(l)<i}^{\sigma^{-1}(i)} q^{-h^-(z_l)} f(z_{\sigma^{-1}(i)}) \prod_{l=\sigma^{-1}(i)+1, \sigma(l)<i}^{l} q^{-h^-(z_l)} \Big\} \Big\rangle_{U_\hbar \mathfrak{g}^{\otimes n}}. \tag{64}$$

Set

$$q(z, w) = q^{2 \sum_i ((T+U)e_i)(z) e^i(w)}. \tag{65}$$

We have $q^{h^-(w)} f(z) q^{-h^-(w)} = q(z, w)^{-1} f(z)$. Using this equality and (62), we identify (64) with

$$\begin{aligned} &\sum_{\sigma \in S_n} \langle \otimes_{i=1}^{n} e[\epsilon_i], \otimes_{i=1}^{n} f(z_{\sigma^{-1}(i)}) \rangle_{U_\hbar \mathfrak{g}^{\otimes n}} \prod_{l<(i), \sigma(l)>\sigma(i)} q(z_i, z_l)^{-1} \\ &= \sum_{\sigma \in S_n} \prod_{i=1}^{n} \epsilon_{\sigma(i)}(z_i) \prod_{l<i, \sigma(l)>\sigma(i)} q(z_i, z_l)^{-1}; \end{aligned}$$

each term of this sum belongs to $\mathbb{C}((z_1))((z_2)) \cdots ((z_n))$. Therefore

$$\begin{aligned} &\langle \prod_{i=1}^{n} e[\epsilon_i], \prod_{i=1}^{n} f[\eta_i] \\ &= \sum_{\sigma \in S_n} \mathrm{res}_{z_n \in S} \cdots \mathrm{res}_{z_1 \in S} \sum_{\sigma \in S_n} \prod_{i=1}^{n} \epsilon_{\sigma(i)}(z_i) \prod_{i=1}^{n} \eta_i(z_i) \\ &\qquad \prod_{l<i, \sigma(l)>\sigma(i)} q(z_i, z_l)^{-1} \omega_{z_1} \cdots \omega_{z_n}, \end{aligned} \tag{66}$$

where $\mathrm{res}_{z \in S} \lambda_z$ means $\sum_{s \in S} \mathrm{res}_s \lambda_z$.

In the case of $U_\hbar \bar{\mathfrak{g}}$, one can lead the similar computation for

$$\langle \prod_{i=1}^{n} f[\eta_i], \prod_{i=1}^{n} e[\epsilon_i] \rangle_{U_\hbar \bar{\mathfrak{g}}}.$$

In that case one uses the identity $(K^+(z), f(w)) = q(z,w)^{-1}$, with $q(z,w)$ defined by (65); the result is the r.h.s. of (66). ∎

Let us define the completion $U_\hbar\mathfrak{g}\bar{\otimes}U_\hbar\mathfrak{g}$ as follows. Let $I_N \subset U_\hbar\mathfrak{g}$ be the left ideal generated by the $x[\epsilon]$, $\epsilon \in \prod_{s\in S} z_s^N\mathbb{C}[[z_s]]$. Define $U_\hbar\mathfrak{g}\bar{\otimes}U_\hbar\mathfrak{g}$ as the inverse limit of the $U_\hbar\mathfrak{g}^{\otimes 2}/I_N\otimes U_\hbar\mathfrak{g}+U_\hbar\mathfrak{g}\otimes I_N$ (where the tensor products are $\hbar$-adically completed). $U_\hbar\mathfrak{g}\bar{\otimes}U_\hbar\mathfrak{g}$ is clearly a completion of $U_\hbar\mathfrak{g}^{\hat{\otimes}2}$. One defines similarly $U_\hbar\mathfrak{g}^{\bar{\otimes}n}$.

Definition 5.1: Let $(\alpha^i), (\alpha_i)$ be dual bases of $U_\hbar\mathfrak{n}_+$ and $U_\hbar\mathfrak{n}_-$. We set

$$F = \sum_i \alpha^i \otimes \alpha_i. \tag{67}$$

From (66) it follows that F belongs to $U_\hbar\mathfrak{g}\bar{\otimes}U_\hbar\mathfrak{g}$.

By (60), we also have

$$F \in 1 + \hbar\sum_i e[\epsilon^i]\otimes f[\epsilon_i] + \sum_{j\geq 2} U_\hbar\mathfrak{n}_+^{[j]}\otimes U_\hbar\mathfrak{n}_-^{[j]}, \tag{68}$$

where $U_\hbar\mathfrak{n}_\pm^{[j]}$ are the degree j homogeneous components of $U_\hbar\mathfrak{n}_\pm$ (where the $e[\epsilon]$ and $f[\epsilon]$ are given homogeneous degree 1).

PROPOSITION 5.2: *Set $q = e^\hbar$ and*

$$\mathcal{R} = q^{D\otimes K}\exp\left(\frac{\hbar}{2}\sum_i h^+[e^i]\otimes h^-[e_i]\right)F,$$

and

$$\bar{\mathcal{R}} = F^{(21)}q^{D\otimes K}\exp\left(\frac{\hbar}{2}\sum_i h^+[e^i]\otimes h^-[e_i]\right);$$

then $\mathcal{R}$ and $\bar{\mathcal{R}}$ are the universal R-matrices of $(U_\hbar\mathfrak{g},\Delta)$ and $(U_\hbar\mathfrak{g}_-,\bar{\Delta})$ (viewed as the doubles of $(U_\hbar\mathfrak{g}_+,\Delta)$and $(U_\hbar\bar{\mathfrak{g}}_+,\bar{\Delta})$), respectively.

Proof: These statements are equivalent to the following ones:

$$\langle id\otimes b_+, \mathcal{R}\rangle_{U_\hbar\mathfrak{g}} = b_+, \quad \langle \mathcal{R}, b_-\otimes id\rangle_{U_\hbar\mathfrak{g}} = b_-,$$

for $b_\pm \in U_\hbar\mathfrak{g}_\pm$, and

$$\langle id\otimes \bar{b}_+, \bar{\mathcal{R}}\rangle_{U_\hbar\bar{\mathfrak{g}}} = \bar{b}_+, \quad \langle \bar{\mathcal{R}}, \bar{b}_-\otimes id\rangle_{U_\hbar\bar{\mathfrak{g}}} = \bar{b}_-,$$

for $\bar{b}_\pm \in U_\hbar\bar{\mathfrak{g}}_\pm$. (Here and later, we will set $\langle id\otimes a, b\otimes c\rangle = \langle a,c\rangle b$, $\langle a\otimes id, b\otimes c\rangle = \langle a,b\rangle c$, etc.) Let us show the first statement.

Assume that b_+ has the form $t_R x_+$, with t_R in $U_\hbar\mathfrak{h}_R$ and x_+ in $U_\hbar\mathfrak{n}_+$. Set $\mathcal{K} = q^{D\otimes K}\exp\left(\frac{\hbar}{2}\sum_i h^+[e^i]\otimes h^-[e_i]\right)$; and set $\mathcal{K} = \sum_j K_j \otimes K_j'$. Then

$$\langle id \otimes b_+, \mathcal{R}\rangle_{U_\hbar\mathfrak{g}} = \sum_{j,i} K_j \alpha^i \langle b_+, K_j'\alpha_i\rangle_{U_\hbar\mathfrak{g}}.$$

Now set $\Delta(b_+) = \sum b_+^{(1)} \otimes b_+^{(2)}$; we have

$$\langle b_+, K_j'\alpha_i\rangle_{U_\hbar\mathfrak{g}} = \sum \langle b_+^{(1)}, K_j'\rangle_{U_\hbar\mathfrak{g}} \langle b_+^{(2)}, \alpha_i\rangle_{U_\hbar\mathfrak{g}}.$$

Only the part of $\Delta(b_+)$ whose first factors are of degree zero contribute to this sum. This part is equal to $\Delta(t_R)(1\otimes x_+)$. Therefore, if $\Delta(t_R) = \sum t_R^{(1)} \otimes t_R^{(2)}$, we have

$$\langle id \otimes b_+, \mathcal{R}\rangle_{U_\hbar\mathfrak{g}} = \sum_{j,i} K_j \alpha^i \sum \langle t_R^{(1)}, K_j'\rangle_{U_\hbar\mathfrak{g}} \langle t_R^{(2)} x_+, \alpha_i\rangle_{U_\hbar\mathfrak{g}};$$

by (62), this is equal to

$$\sum_{j,i} K_j \alpha^i \sum \langle t_R, K_j'\rangle_{U_\hbar\mathfrak{g}} \langle x_+, \alpha_i\rangle_{U_\hbar\mathfrak{g}} = \sum_j K_j x_+ \langle t_R, K_j'\rangle_{U_\hbar\mathfrak{g}}.$$

One easily checks that $\sum_j K_j \langle t_R, K_j'\rangle_{U_\hbar\mathfrak{g}} = t_R$. Therefore, the last sum is equal to b_+.

The proof of the other statements is similar. ∎

LEMMA 5.2: *$\mathcal{K}$ satisfies the cocycle identity*

$$\mathcal{K}^{12}(\bar\Delta\otimes 1)(\mathcal{K}) = \mathcal{K}^{23}(1\otimes\bar\Delta)(\mathcal{K}).$$

Proof: $(\mathcal{B}^0, \bar\Delta)$ is a Hopf subalgebra of $(U_\hbar\mathfrak{g}, \bar\Delta)$. It is easy to check that $(U_\hbar\mathfrak{h}_R, \bar\Delta)$ and $(U_\hbar\mathfrak{h}_\Lambda, \bar\Delta')$ are dual Hopf algebra, and that the double of $(U_\hbar\mathfrak{h}_R, \bar\Delta)$ is $(\mathcal{B}^0, \bar\Delta)$. Moreover, $\mathcal{K}$ represents the identity pairing between these algebras. The identities of the Lemma are then consequences of the quasi-triangular identities. ∎

LEMMA 5.3: *$\mathcal{K}$ conjugates the coproducts $\bar\Delta$ and Δ', that is*

$$\Delta'(x) = \mathcal{K}\bar\Delta(x)\mathcal{K}^{-1} \tag{69}$$

for any x in $U_\hbar\mathfrak{g}$.

Proof: Let us first prove this identity for x in $\mathcal{B}^0$. The R-matrix identity for $(\mathcal{B}^0, \bar\Delta)$ says that $\bar\Delta'(x) = \mathcal{K}\bar\Delta(x)\mathcal{K}^{-1}$ for x in $\mathcal{B}^0$. On the other hand, the restrictions of Δ and $\bar\Delta$ to $\mathcal{B}^0$ coincide. This proves (69) in this case.

Let us now treat the case where $x = e(z)$. Set $\mathcal{K}_0 = \exp(\frac{\hbar}{2}\sum_i h^+[e^i] \otimes h^-[e_i])$ and $\mathcal{K}_D = q^{D\otimes K}$. Then $\mathcal{K} = \mathcal{K}_D\mathcal{K}_0$.

We have $\bar\Delta(e(z)) = (e \otimes q^{-h^-})(q^{K_2\partial}z) + 1 \otimes e(z)$. Then we have

$$[\sum_i h^+[e^i] \otimes h^-[e_i], 1 \otimes e(z)] = (T+U)(h^+(q^{K_2\partial}z))_\Lambda \otimes e(z),$$

so that

$$\mathcal{K}_0(1 \otimes e(z))\mathcal{K}_0^{-1} = \exp(\hbar\sum_i h^+[e^i]((T+U)(q^{K_2\partial}e_i))_\Lambda(z)) \otimes e(z),$$

and

$$\mathcal{K}_D\mathcal{K}_0(1 \otimes e(z))\mathcal{K}_0^{-1}\mathcal{K}_D^{-1} = \exp(\hbar\sum_i h^+[q^{K_2\partial}e^i]((T+U)(q^{K_2\partial}e_i))_\Lambda(z)) \otimes e(z)$$
$$= \exp(\hbar\sum_i h^+[e^i](T+U)e_i(z)) \otimes e(z)$$
$$= q^{(T+U)h^+}(z) \otimes e(z). \tag{70}$$

On the other hand,

$$\mathcal{K}_0(e(z) \otimes 1)\mathcal{K}_0^{-1} = e(z) \otimes q^{h^-(z)}, \tag{71}$$

and

$$[h^-[e_i], h^-(z)] = \frac{2}{\hbar}\left((q^{-K_2\partial}(T(q^{K_2\partial}e_i)_R))(z) + Ue_i(z) - (q^{-K_2\partial}U((q^{K_2\partial}e_i)_\Lambda))(z)\right),$$

so that

$$[\frac{\hbar}{2}\sum_i h^+[e^i] \otimes h^-[e_i], 1 \otimes h^-(z)]$$
$$= (q^{-K_2\partial}T(q^{K_2\partial}h^+)_R)(z) + Uh^+(z) - q^{-K_2\partial}(U(q^{K_2\partial}h^+)_\Lambda)(z) \otimes 1,$$

so that

$$\mathcal{K}_0(1 \otimes h^-(q^{K_2\partial}z))\mathcal{K}_0^{-1}$$
$$= 1 \otimes q^{K_2\partial}h^-(z) + \left(T(q^{K_2\partial}h^+)_R(z) + Uh^+(q^{K_2\partial}z) - (U(q^{K_2\partial}h^+)_\Lambda)(z)\right) \otimes 1$$

and

$$\mathcal{K}_0(1 \otimes q^{-h^-(q^{K_2\partial}z)})\mathcal{K}_0^{-1}$$
$$= \exp(-\hbar\left(T(q^{K_2\partial}h^+)_R(z) + Uh^+(q^{K_2\partial}z) - (U(q^{K_2\partial}h^+)_\Lambda)(z)\right)) \otimes q^{-h^-(q^{K_2\partial}z)}. \tag{72}$$

Finally, the product of (71) and (72) gives

$$\begin{aligned}(73)\qquad &\mathcal{K}_0((e\otimes q^{-h^-})(q^{K_2\partial}z))\mathcal{K}_0^{-1}\\ &=\exp(-\hbar\,(T(q^{K_2\partial}h^+)_R(z)+Uh^+(q^{K_2\partial}z)\\ &\qquad -(U(q^{K_2\partial}h^+)_\Lambda)(z)))e(q^{K_2\partial}z)\otimes 1.\end{aligned}$$

On the other hand, we have

$$(D+\partial_z-Ah^+(z))e(z)=e(z)(D+\partial_z)$$

(an identity of differential operators). Therefore we have

$$q^{K_2(D+\partial_z-Ah^+(z))}e(z)=e(z)q^{K_2(D+\partial_z)}. \tag{74}$$

We are now in position to apply the following identity:

Lemma 5.4: *Let L,f be elements of a Lie algebra, such that all $\mathrm{ad}^k(L)(f)$ commute with f. We have the identity, in the associated formal group*

$$\alpha^{L+f}=\alpha^L\exp\left(\frac{1-\alpha^{-\,\mathrm{ad}\,L}}{\mathrm{ad}\,L}(f)\right),$$

where $\alpha=e^{\hbar k}$, k scalar and $\hbar$ a formal parameter.

Proof: Enlarge the Lie algebra by adjoining to it an element F, such that $[L,F]=f$. We have in the associated formal group,

$$\begin{aligned}\alpha^{L+f}&=\alpha^{\mathrm{Ad}(e^{-F})(L)}=\mathrm{Ad}(e^{-F})(\alpha^L)=\alpha^L[(\alpha^{-\,\mathrm{ad}(L)}(e^{-F}))e^F]\\ &=\alpha^L\exp((1-\alpha^{-\,\mathrm{ad}(L)})(F))=\alpha^L\exp\left(\frac{1-\alpha^{-\,\mathrm{ad}(L)}}{\mathrm{ad}(L)}(f)\right).\end{aligned}$$ ∎

Set in (74), $\alpha=q^{K_2}$, $L=D+\partial_z$ and $f=-Ah^+(z)$. Then we need to compute

$$\frac{1-q^{-K_2\,\mathrm{ad}(D+\partial_z)}}{\mathrm{ad}(D+\partial_z)}(-Ah^+)(z).$$

For this, we show:

Lemma 5.5: *Let B be some linear operator from $\Lambda[[\hbar]]$ to $R[[\hbar]]$, such that $Be_i\to 0$ when $i\to\infty$, then*

(1) $\mathrm{ad}(D+\partial_z)(Bh^+(z))=Ch^+(z)$, *where* $C=\partial\circ B-B\circ pr_\Lambda\circ\partial$,
(2) $\mathrm{Ad}(\alpha^{D+\partial_z})(Bh^+(z))=Gh^+(z)$, *where* $G=\alpha^\partial\circ B\circ pr_\Lambda\circ\alpha^{-\partial}$, *and* $\alpha=e^{\hbar k}$, *k scalar,*

(3) *assume that* $B = \partial \circ E - E \circ pr_\Lambda \circ \partial$, *with* E *a linear operator from* $\Lambda[[\hbar]]$ *to* $k[[\hbar]]$, *then*

$$\frac{\alpha^{\mathrm{ad}(D+\partial_z)} - 1}{\mathrm{ad}(D + \partial_z)} \left(Bh^+(z)\right) = Fh^+(z),$$

where $F = \alpha^{\partial} \circ E \circ pr_\Lambda \circ \alpha^{\partial} - E$.

Proof: (1) is obvious. (2) is obtained by first computing

$$\overset{k}{\mathrm{ad}}(D + \partial_z)(Bh^+(z)[e^i]),$$

using (1). We again use this expression to obtain (3). ∎

Applying Lemma 5.5 with $B = A$, $E = \hbar(T + U)$, $\alpha = q^{-K_2}$, we get

$$\frac{1 - q^{-K_2\,\mathrm{ad}(D+\partial_z)}}{\mathrm{ad}(D + \partial_z)}(-Ah^+)(z). = \frac{q^{-K_2\,\mathrm{ad}(D+\partial_z)} - 1}{\mathrm{ad}(D + \partial_z)}(Ah^+)(z). = Fh^+(z),$$

with $F = \hbar(q^{-K_2\partial} \circ (T + U) \circ pr_\Lambda - (T + U))$. Therefore Lemma 5.4 gives

$$\begin{aligned} q^{K_2(D+\partial_z - Ah^+(z))} &= q^{K_2(D+\partial_z}\exp(\frac{1 - q^{-K_2\,\mathrm{ad}(D+\partial_z)}}{\mathrm{ad}(D + \partial_z)}(-Ah^+)(z)) \\ &= q^{K_2(D+\partial_z}\exp([q^{-K_2\partial} \circ \hbar(T + U) \circ pr_\Lambda - \hbar(T + U)]h^+(z)). \end{aligned}$$

Finally, (74) gives

$$q^{K_2(D+\partial_z)}\exp([q^{-K_2\partial} \circ \hbar(T + U) \circ pr_\Lambda - \hbar(T + U)]h^+(z))e(z) = e(z)q^{K_2(D+\partial_z)},$$

so that

$$q^{-K_2D}e(z)q^{K_2D} = q^{K_2\partial_z}\{\exp([q^{-K_2\partial} \circ \hbar(T + U) \circ pr_\Lambda - \hbar(T + U)]h^+(z))e(z)\},$$

which coincides with the r.h.s. of (73). Therefore

$$\mathcal{K}_D\mathcal{K}_0((e \otimes q^{-h^-})(q^{K_2\partial}z))(\mathcal{K}_D\mathcal{K}_0)^{-1} = e(z) \otimes 1. \tag{75}$$

Adding up (70) and (75), we get

$$\Delta'(e(z)) = \mathcal{K}\bar{\Delta}(e(z))\mathcal{K}^{-1}.$$

The proof is similar when $e(z)$ is replaced by the case of $f(z)$. ∎

PROPOSITION 5.3: *F satisfies the cocycle identity*

$$F^{12}(\Delta \otimes 1)(F) = F^{23}(1 \otimes \Delta)(F). \tag{76}$$

Proof: The quasi-triangularity identities for $\bar{\mathcal{R}}$ imply that

$$\bar{\mathcal{R}}^{12}(\bar{\Delta}\otimes 1)(\bar{\mathcal{R}}) = \bar{\mathcal{R}}^{23}(1\otimes\bar{\Delta})(\bar{\mathcal{R}}).$$

Therefore, we have

$$F^{21}\mathcal{K}^{12}(\bar{\Delta}\otimes 1)(F^{21})(\bar{\Delta}\otimes 1)(K) = F^{32}\mathcal{K}^{23}(1\otimes\bar{\Delta})(F^{21})(1\otimes\bar{\Delta})(K).$$

By Lemma 5.3, it follows that

$$F^{21}(\Delta'\otimes 1)(F^{21})\mathcal{K}^{12}(\bar{\Delta}\otimes 1)(\mathcal{K}) = F^{32}(1\otimes\Delta')(F^{21})\mathcal{K}^{23}(1\otimes\bar{\Delta})(\mathcal{K}).$$

Lemma 5.2 then implies that

$$F^{21}(\Delta'\otimes 1)(F^{21}) = F^{32}(1\otimes\Delta')(F^{21}),$$

which is the same as (76), up to permutation of factors. ∎

PROPOSITION 5.4: *Δ and $\bar{\Delta}$ are conjugated by F: we have $\bar{\Delta}(z) = F\Delta(z)F^{-1}$, for any x in $U_\hbar\mathfrak{g}$.*

Proof: We know that $\Delta' = \mathcal{R}\Delta\mathcal{R}^{-1}$. Since $\mathcal{R} = \mathcal{K}F$, it follows that $F\Delta F^{-1} = \mathcal{K}^{-1}\Delta'\mathcal{K}$. But by Lemma 5.3, $\mathcal{K}^{-1}\Delta'\mathcal{K}$ coincides with $\bar{\Delta}$. ∎

Remark 5.1: It is a general principle in R-matrix computations (see [14]) that factors of the R-matrix are also twists relating quantizations of conjugated Manin triples. We see that this principle also holds in our situation.

5.1 ORTHOGONALS OF $\mathcal{B}_R^{\pm}$.

PROPOSITION 5.5:

(1) *The orthogonal of $\mathcal{B}_R^-$ in $\mathcal{B}^+$ for $\langle,\rangle_{U_\hbar\mathfrak{g}}$ is the span $\mathfrak{n}_+(R)\mathcal{B}^+$ of all $e[r]e[\epsilon_1]\cdots e[\epsilon_p]$, $p\ge 0$, $\epsilon_i\in k$, $r\in R$;*

(2) *the orthogonal of $\mathcal{B}_R^+$ in $\mathcal{B}^-$ for $\langle,\rangle_{U_\hbar\mathfrak{g}}$ is the span $\mathcal{B}^-\mathfrak{n}_-(R)$ of all*

$$f[\eta_1]\cdots f[\eta_p]f[r],\quad p\ge 0,\ \eta_i\in k,\ r\in R.$$

Proof: Let us compute $\langle e[r]e[\epsilon_1]\cdots e[\epsilon_p], f[r_1]\cdots f[r_{p+1}]\rangle_{U_\hbar\mathfrak{g}}$, $\epsilon_i\in k$, $r, r_i\in R$. Expand it as

$$\langle e[r]\otimes e[\epsilon_1]\cdots e[\epsilon_p], \Delta'(f[r_1]\cdots f[r_{p+1}])\rangle_{U_\hbar\mathfrak{g}^{\hat{\otimes}2}}$$

(recall that Δ' is Δ composed with the exchange of factors). According to the proof of Prop. 4.4, $\Delta'(f[r_1]\cdots f[r_{p+1}])$ can be decomposed as a sum of terms, the first factors all of which are of the form $f[r'_1]\cdots f[r'_s]$, $r'_i \in R$. Since $\langle e[r], f[r'_1]\cdots f[r'_s]\rangle_{U_\hbar\mathfrak{g}}$ is always zero (either because $s \neq 1$ of by isotropy of R),

$$\langle e[r]e[\epsilon_1]\cdots e[\epsilon_p], f[r_1]\cdots f[r_{p+1}]\rangle_{U_\hbar\mathfrak{g}} = 0.$$

This shows that $\mathfrak{n}_+(R)\mathcal{B}^+ \subset (\mathcal{B}_R^-)^\perp$.

To show that $\mathfrak{n}_+(R)\mathcal{B}^+$ is the whole of $(\mathcal{B}_R^-)^\perp$, let us consider the classical limit of the situation. $\mathcal{B}_R^- \subset \mathcal{B}^-$ is a flat deformation of the inclusion of symmetric algebras $S^*(R) \subset S^*(k)$. On the other hand, the inclusion $\mathfrak{n}_+(R)\mathcal{B}^+ \subset \mathcal{B}^+$ is a flat deformation of $S^*(k)R \subset S^*(k)$. Finally, let $\mathcal{B}_p^i$ be the completion of the span of all $x[\epsilon_1]\cdots x[\epsilon_p], \epsilon_i \in k$, $x = e, f$ for $i = +, -$, and multiply $\langle , \rangle_{U_\hbar\mathfrak{g}}$ by $\hbar^p$ on $\mathcal{B}_p^+ \times \mathcal{B}_q^-$. Then the resulting pairing is a deformation of the direct sum of the symmetric powers of the pairing $\langle , \rangle_k$. Since the orthogonal of $S^*(R)$ for this pairing is $S^*(k)R$ (because R is maximal isotropic, see Lemma 7.2), the orthogonal of $\mathcal{B}_R^-$ cannot be larger than $\mathfrak{n}_+(R)\mathcal{B}^+$. This finally shows (1).

Let us pass to the proof of (2). Let us compute

$$\langle e[r_1]\cdots e[r_{p+1}], f[\eta_1]\cdots f[\eta_p]f[r]\rangle_{U_\hbar\mathfrak{g}}, \quad r, r_i \in R, \quad \eta_i \in k.$$

Expand it as $\langle\Delta(e[r_1]\cdots e[r_{p+1}]), f[\eta_1]\cdots f[\eta_p]\otimes f[r]\rangle_{U_\hbar\mathfrak{g}^{\hat\otimes 2}}$. From the proof of Proposition 4.4 follows that $\Delta(e[r_1]\cdots e[r_{p+1}])$ can be decomposed as a sum of terms, the second factors all of which lie in the algebra generated by the $e[r], h^+[r'], r, r' \in R$.

Further decompose each of these second factors as a sum of terms of the form

$$e[r''_1]\cdots e[r''_s]h^+[\bar r_1]\cdots h^+[\bar r_t], \quad r''_i, \bar r_i \in R.$$

The pairing of this with $f[r]$ gives

$$\langle e[r''_1]\cdots e[r''_s]\otimes h^+[\bar r_1]\cdots h^+[\bar r_t], \Delta'(f[r])\rangle_{U_\hbar\mathfrak{g}^{\hat\otimes 2}}.$$

$\Delta'(f[r])$ is equal to the sum of $1\otimes f[r]$ and of a sum of terms, the first factors of which are either 1 or of the form $f[\rho], \rho \in R$.

Since $\langle e[r''_1]\cdots e[r''_s],\rangle f[\rho]\rangle_{U_\hbar\mathfrak{g}} = 0$ (either by degree reasons if $s \neq 1$ or by isotropy of R), the only possibly non-trivial contribution is that of

$$\langle e[r''_1]\cdots e[r''_s]\otimes h^+[\bar r_1]\cdots h^+[\bar r_t], 1\otimes f[r]\rangle_{U_\hbar\mathfrak{g}^{\hat\otimes 2}};$$

but the pairing of $f[r]$ with any $h^+[\bar r_1]\cdots h^+[\bar r_t]$ is zero.

This shows that $\langle e[r_1]\cdots e[r_{p+1}], f[\eta_1]\cdots f[\eta_p]f[r]\rangle_{U_\hbar\mathfrak{g}} = 0$, for any $r, r_i \in R, \eta_i \in k$, so that $\mathcal{B}^-\mathfrak{n}_-(R) \subset (\mathcal{B}_R^+)^\perp$.

The proof that $\mathcal{B}^-\mathfrak{n}_-(R)$ is actually equal to $(\mathcal{B}_R^+)^\perp$ is similar to the argument used in the proof of (1). ∎

6. Quasi-Hopf structures

6.1 FACTORIZATION OF F. We now recall our aim. We would like to decompose F defined in (67) as a product

$$F_2F_1, \quad \text{with} \quad F_1 \in U_\hbar\mathfrak{g}\hat{\otimes}U_\hbar\mathfrak{g}_R, \quad F_2 \in U_\hbar\mathfrak{g}_R\hat{\otimes}U_\hbar\mathfrak{g}. \tag{77}$$

The interest of this decomposition lies in the following proposition.

PROPOSITION 6.1: *For any decomposition (77), the map* $\mathrm{Ad}(F_1)\circ\Delta$ *defines an algebra morphism from* $U_\hbar\mathfrak{g}_R$ *to* $U_\hbar\mathfrak{g}_R\hat{\otimes}U_\hbar\mathfrak{g}_R$ *(where the tensor product is completed over* $\mathbb{C}[[\hbar]]$*).*

Proof: Since $\bar{\Delta} = \mathrm{Ad}(F)\circ\Delta$, we have

$$\mathrm{Ad}(F_1)\circ\Delta = \mathrm{Ad}(F_2^{-1})\circ\bar{\Delta}.$$

The first map sends $U_\hbar\mathfrak{g}_R$ to $U_\hbar\mathfrak{g}\hat{\otimes}U_\hbar\mathfrak{g}_R$, and the second one to $U_\hbar\mathfrak{g}_R\hat{\otimes}U_\hbar\mathfrak{g}$, by Proposition 4.4 and (77); so both maps send $U_\hbar\mathfrak{g}_R$ to

$$(U_\hbar\mathfrak{g}\hat{\otimes}U_\hbar\mathfrak{g}_R)\cap(U_\hbar\mathfrak{g}_R\hat{\otimes}U_\hbar\mathfrak{g}) = U_\hbar\mathfrak{g}_R\hat{\otimes}U_\hbar\mathfrak{g}_R. \quad ∎$$

Let us now try to decompose F according to (77). Let (m_i), resp. (m_i') be a basis of $U_\hbar\mathfrak{g}$ as a left, resp. right $U_\hbar\mathfrak{g}_R$-module. Assume $m_0 = m_0' = 1$. Due to the form of F_1 and F_2, we have decompositions

$$F_2 = \sum_i (1\otimes m_j')F_2^{(j)}, \quad F_1 = \sum_i F_1^{(i)}(m_i\otimes 1), \quad F_1^{(i)}, F_2^{(j)} \in U_\hbar\mathfrak{g}_R^{\hat{\otimes}2}.$$

It follows that we have

$$F = \sum_i F_2F_1^{(i)}(m_i\otimes 1) = \sum_j (1\otimes m_j')F_2^{(j)}F_1. \tag{78}$$

Let now Π, resp. Π' be the left, resp. right $U_\hbar\mathfrak{g}_R$-module morphisms from $U_\hbar\mathfrak{g}$ to $U_\hbar\mathfrak{g}_R$, such that $\Pi(m_i) = 0$ for $i \neq 0$, $\Pi(1) = 1$, and $\Pi'(m_i') = 0$ for $i \neq 0$, $\Pi'(1) = 1$.

From (78) follows that we should have

$$F_2F_1^{(0)} = (\Pi\otimes 1)F, \quad F_2^{(0)}F_1 = (1\otimes\Pi')F. \tag{79}$$

We may and will assume that m_i, resp. m'_i contains a basis of $\mathcal{B}^+$ as a left $\mathcal{B}^+_R$-module, resp. of $\mathcal{B}^-$ as a right $\mathcal{B}^-_R$-module. Then, Π maps $\mathcal{B}^+$ to $\mathcal{B}^+_R$, and Π' maps $\mathcal{B}^-$ to $\mathcal{B}^-_R$. It follows that $[(\Pi\otimes 1)F]^{-1}F[(1\otimes\Pi')F]^{-1}$ belongs to $\mathcal{B}^+\hat{\otimes}\mathcal{B}^-$.

Equation (79) determines the possible values of F_1 and F_2, up to right, resp. left multiplication by elements of $U_\hbar\mathfrak{g}_R^{\hat{\otimes}2}$.

PROPOSITION 6.2: *Let* $F_{\Pi,\Pi'} = [(\Pi\otimes 1)F]^{-1}F[(1\otimes\Pi')F]^{-1}$; *then*

$$F_{\Pi,\Pi'} \in U_\hbar\mathfrak{g}_R^{\hat{\otimes}2}. \tag{80}$$

Proof: Since $(\Pi\otimes 1)F \in U_\hbar\mathfrak{g}_R\hat{\otimes}U_\hbar\mathfrak{g}$, and $(1\otimes\Pi')F \in U_\hbar\mathfrak{g}\hat{\otimes}U_\hbar\mathfrak{g}_R$, (80) is equivalent to showing that

$$F^{-1}[(\Pi\otimes 1)F] \in \mathcal{B}^+\hat{\otimes}\mathcal{B}^-_R, \quad [(1\otimes\Pi')F]F^{-1} \in \mathcal{B}^+_R\hat{\otimes}\mathcal{B}^-. \tag{81}$$

By Proposition 5.2, the universal R-matrices of $U_\hbar\mathfrak{g}$ and $U_\hbar\bar{\mathfrak{g}}$, $\mathcal{R}$ and $\bar{\mathcal{R}}$ are such that $\mathcal{R} \in (U_\hbar\mathfrak{g}_R\hat{\otimes}U_\hbar\mathfrak{g})F$ and $\bar{\mathcal{R}}^{21} \in F(U_\hbar\mathfrak{g}\hat{\otimes}U_\hbar\mathfrak{g}_R)$.

Since Π, resp. Π' is a left, resp. right $U_\hbar\mathfrak{g}_R$-module morphism, it follows that

$$F^{-1}[(\Pi\otimes 1)F] = \mathcal{R}^{-1}[(\Pi\otimes 1)\mathcal{R}] \quad \text{and} \quad [(1\otimes\Pi')F]F^{-1} = [(1\otimes\Pi')\bar{\mathcal{R}}^{21}](\bar{\mathcal{R}}^{21})^{-1}. \tag{82}$$

We will need the following result.

LEMMA 6.1:

(1) *Let S' denote the skew antipode of $U_\hbar\mathfrak{g}$, then for any $x \in \mathcal{B}^+$,*

$$\langle\mathcal{R}^{-1}, id\otimes x\rangle_{U_\hbar\mathfrak{g}} = S'(x).$$

(2) *Recall $\bar{S}$ is the antipode of $U_\hbar\bar{\mathfrak{g}}$. For any $y \in \mathcal{B}_-$, we have*

$$\langle\bar{\mathcal{R}}^{-1}, id\otimes y\rangle_{U_\hbar\bar{\mathfrak{g}}} = \bar{S}(y).$$

Proof: Since $(\Delta\otimes 1)(\mathcal{R}^{-1}) = (\mathcal{R}^{-1})^{23}(\mathcal{R}^{-1})^{13}$,

$$\sigma' : \mathcal{B}_+ \to U_\hbar\mathfrak{g}_+, \quad x \mapsto \langle\mathcal{R}^{-1}, id\otimes x\rangle_{U_\hbar\mathfrak{g}}$$

is an algebra morphism from $\mathcal{B}^+$ to $U_\hbar\mathfrak{g}_+^{opp}$ (that is, $U_\hbar\mathfrak{g}_+$ with the opposite multiplication). Since S' is also an algebra morphism from $\mathcal{B}^+$ to $U_\hbar\mathfrak{g}_+^{opp}$, it suffices to check that S' and σ' coincide on the generators of $\mathcal{B}^+$.

From (19) follows that $S'(e(z)) = -e(z)q^{-((T+U)h^+)(z)}$, on the other hand

$$\begin{aligned}\sigma'(e(z)) &= \langle \mathcal{R}^{-1}, id \otimes e(z)\rangle_{U_\hbar\mathfrak{g}} \\ &= \langle (1 - \hbar \sum_{i\in\mathbb{Z}} e[\epsilon^i] \otimes f[\epsilon_i] + \cdots) e^{-\frac{\hbar}{2}\sum_{j\in\mathbb{N}} h^+[e^j]\otimes h^-[e_j]}, id \otimes e(z)\rangle_{U_\hbar\mathfrak{g}} \\ &= \langle (-\hbar \sum_{i\in\mathbb{Z}} e[\epsilon^i] \otimes f[\epsilon_i]) e^{-\frac{\hbar}{2}\sum_{j\in\mathbb{N}} h^+[e^j]\otimes h^-[e_j]}, id \otimes e(z)\rangle_{U_\hbar\mathfrak{g}} \\ &= \langle (-\hbar) \sum_{i\in\mathbb{Z}} e[\epsilon^i]^{[1]} f[\epsilon_i]^{[2]} e^{-\frac{\hbar}{2}\sum_{j\in\mathbb{N}} h^+[e^j]^{[1]} h^-[e_j]^{[3]}}, \\ &\qquad\qquad id \otimes e(z) \otimes q^{((T+U)h^+)(z)}\rangle_{U_\hbar\mathfrak{g}} \\ &= -e(z)q^{-((T+U)h^+)(z)}.\end{aligned}$$

In the r.h.s. of the first equality, the notation means that we perform the pairing of the second factors of a decomposition of $\mathcal{R}^{-1}$ with $e(z)$, so this r.h.s. belongs to $U_\hbar\mathfrak{g}_R$; the third equality is because $e(z)$ cannot have a nontrivial pairing with a product of zero or more than two $f[\epsilon_i]$'s; in the fourth equality, we used the notation $a^{[i]}$ for $1^{\otimes(i-1)} \otimes a \otimes 1^{\otimes(3-i)}$; the last equality follows from

$$\langle (-\hbar) \sum_{i\in\mathbb{Z}} e[\epsilon^i] \otimes f[\epsilon_i], id \otimes e(z)\rangle_{U_\hbar\mathfrak{g}} = -e(z),$$

and

$$\langle e^{-\frac{\hbar}{2}\sum_{j\in\mathbb{N}} h^+[e^j]\otimes h^-[e_j]}, id \otimes q^{((T+U)h^+)(z)}\rangle_{U_\hbar\mathfrak{g}} = q^{-((T+U)h^+)(z)},$$

which follow from direct calculation. This finally shows the first part of the lemma. The proof of the second part is similar. ∎

Let $r \in R$, $\phi \in \mathcal{B}^+$, and let us compute $\langle F^{-1}[(\Pi \otimes 1)F], id \otimes e[r]\phi\rangle_{U_\hbar\mathfrak{g}}$. We find

$$\begin{aligned}&\langle F^{-1}[(\Pi \otimes 1)F], id \otimes e[r]\phi\rangle_{U_\hbar\mathfrak{g}} \\ &= \langle \mathcal{R}^{-1}[(\Pi \otimes 1)\mathcal{R}], id \otimes e[r]\phi\rangle_{U_\hbar\mathfrak{g}} \\ &= \sum \langle \mathcal{R}^{-1}, id \otimes (e[r]\phi)^{(1)}\rangle_{U_\hbar\mathfrak{g}} \Pi\left(\langle \mathcal{R}, id \otimes (e[r]\phi)^{(2)}\rangle_{U_\hbar\mathfrak{g}}\right) \\ &= \sum S'(e[r]^{(1)}\phi^{(1)}) \Pi(e[r]^{(2)}\phi^{(2)}),\end{aligned} \tag{83}$$

where the first equality follows from (82), and the second one from the Hopf algebra pairing rules. To show the third one, we remark that $(e[r]\phi)^{(1)}$ belongs to $\mathcal{B}^+$, and apply to it Lemma 6.1, 1).

The r.h.s. of (83) is then $\sum S'(e[r]^{(1)}\phi^{(1)})\Pi(e[r]^{(2)}\phi^{(2)})$, but since $e[r]^{(2)} \in U_\hbar\mathfrak{g}_R$ (see Proposition 4.4), and Π is a left $U_\hbar\mathfrak{g}_R$-module morphism, this is equal to

$$\sum S'(e[r]^{(1)}\phi^{(1)}) e[r]^{(2)} \Pi(\phi^{(2)})$$

or

$$\sum S'(\phi^{(1)})S'(e[r]^{(1)})e[r]^{(2)}\Pi(\phi^{(2)})$$

(because S' is an algebra anti-automorphism of $U_\hbar\mathfrak{g}$). Since $\sum S'(e[r]^{(1)})e[r]^{(2)} = 0$, the r.h.s of (83) is equal to zero. By Lemma 5.1 (1), it then follows that

$$F^{-1}[(\Pi\otimes 1)F] \in \mathcal{B}^+\hat{\otimes}\mathcal{B}^-_R,$$

which is the first part of (81).

The proof of the second part of (81) is similar: let $r \in R$, $\psi \in \mathcal{B}^+$, and let us compute $\langle[(1\otimes\Pi')F]F^{-1}, \psi f[r]\otimes id\rangle_{U_\hbar\mathfrak{g}}$. The Hopf algebra notation employed now refers to $U_\hbar\bar{\mathfrak{g}}$.

We find

$$\begin{aligned}
\langle[(1\otimes\Pi')F]F^{-1}, \psi f[r]\otimes id\rangle_{U_\hbar\mathfrak{g}} &= [(1\otimes\Pi')F]F^{-1}, \psi f[r]\otimes id\rangle_{U_\hbar\bar{\mathfrak{g}}}\\
&= \langle[(1\otimes\Pi')\bar{\mathcal{R}}^{21}](\bar{\mathcal{R}}^{21})^{-1}, \psi f[r]\otimes id\rangle_{U_\hbar\bar{\mathfrak{g}}}\\
&= \sum \Pi'\left(\langle\bar{\mathcal{R}}^{21}, (\psi f[r])^{(1)}\otimes id\rangle_{U_\hbar\bar{\mathfrak{g}}}\right)\\
&\qquad \langle(\bar{\mathcal{R}}^{21})^{-1}, (\psi f[r])^{(2)}\otimes id\rangle_{U_\hbar\bar{\mathfrak{g}}}\\
&= \sum \Pi'((\psi f[r])^{(1)})\bar{S}((\psi f[r])^{(2)})
\end{aligned} \tag{84}$$

where the first equality follows from Proposition 5.1, the second equality follows from (82), and third one by the Hopf algebra pairing rules. To show the last one, we remark that $(\psi f[r])^{(2)}$ belongs to $\mathcal{B}^-$, and apply to it Lemma 6.1 (2).

The r.h.s. of (84) is then $\sum \Pi'(\psi^{(1)}f[r]^{(1)})S(\psi^{(2)}f[r]^{(2)})$, but since $f[r]^{(1)} \in U_\hbar\mathfrak{g}_R$ (see Lemma 4.4), and Π' is a right $U_\hbar\mathfrak{g}_R$-module morphism, this is equal to

$$\sum \Pi'(\psi^{(1)})f[r]^{(1)}\bar{S}(\psi^{(2)}f[r]^{(2)})$$

or

$$\sum \Pi'(\psi^{(1)})f[r]^{(1)}S(f[r]^{(2)})\bar{S}(\psi^{(2)})$$

(because $\bar{S}$ is an algebra anti-automorphism of $U_\hbar\mathfrak{g}$). Since $\sum f[r]^{(1)}S(\bar{f}[r]^{(2)}) = 0$, the r.h.s of (84) is equal to zero. By Lemma 5.5 (2), it then follows that

$$[(1\otimes\Pi')F]F^{-1} \in \mathcal{B}^+_R\hat{\otimes}\mathcal{B}^-,$$

that is the second part of (81). ∎

PROPOSITION 6.3: *Any decomposition of F according to (77) is of the form*

$$F_2 = [(\Pi\otimes 1)F]b, \quad F_1 = a[(1\otimes\Pi')F],$$

with $a, b \in U_\hbar\mathfrak{g}_R^{\hat{\otimes}2}$, such that $ab = F_{\Pi,\Pi'}$.

Proof: Clear. ∎

COROLLARY 6.1: *For any left $\mathcal{B}_R^-$-module morphism $\bar\Pi$ from $\mathcal{B}^-$ to $\mathcal{B}_R^-$, and any right $\mathcal{B}_R^+$-module morphism $\bar\Pi'$ from $\mathcal{B}^+$ to $\mathcal{B}_R^+$, such that $\bar\Pi(1) = 1, \bar\Pi'(1) = 1$, we have*

$$[(\Pi\otimes 1)F]^{-1}[(\bar\Pi\otimes 1)F] \in U_\hbar\mathfrak{g}_R^{\hat\otimes 2}, \quad [(1\otimes\bar\Pi')F][(1\otimes\Pi')F]^{-1} \in U_\hbar\mathfrak{g}_R^{\hat\otimes 2}.$$

Proof: This follows from the fact that any $\bar\Pi$, $\bar\Pi'$ yield solutions to (77), and from the classification of all such solutions in Proposition 6.3. ∎

CONVENTION 6.1: *The expansion in $\hbar$ of F is $1+\hbar f + o(\hbar)$, in the notation of Lemma 1.1. We may assume that $\Pi(f[e_i]) = 0$, $\Pi'(e[e_i]) = 0$, for all $i \in \mathbb{N}$; this implies that $(1\otimes\Pi')F = 1+\hbar f_1 + o(\hbar)$, and $(\Pi\otimes 1)F = 1+\hbar f_2+o(\hbar)$, so that $F_{\Pi,\Pi'} = 1+o(\hbar)$. In what follows, we will only consider solutions of (77), such that $F_1 = 1+\hbar f_1 + o(\hbar)$, and $F_2 = 1+\hbar f_2+o(\hbar)$; equivalently, the a and b of Proposition 6.3 have the form $1+o(\hbar)$.* ∎

6.2 QUASI-HOPF STRUCTURES ON $U_\hbar\mathfrak{g}_R$ AND $U_\hbar\mathfrak{g}$.

Let us choose a solution (F_1, F_2) of (77), satisfying the above requirement. Consider the algebra morphism $\Delta_R : U_\hbar\mathfrak{g} \to U_\hbar\mathfrak{g}^{\hat\otimes 2}$, defined as

$$\Delta_R = \mathrm{Ad}(F_1)\circ\Delta = \mathrm{Ad}(F_2^{-1})\circ\bar\Delta; \tag{85}$$

define

$$\Phi = F_1^{23}(1\otimes\Delta)(F_1)[F_1^{12}(\Delta\otimes 1)(F_1)]^{-1}. \tag{86}$$

PROPOSITION 6.4: *Φ belongs to $U_\hbar\mathfrak{g}_R^{\hat\otimes 3}$, and even to $\mathcal{B}_R^+\otimes U_\hbar\mathfrak{g}_R\otimes\mathcal{B}_R^-$.*

Proof: (86) makes it clear that Φ belongs to $U_\hbar\mathfrak{g}^{\hat\otimes 2}\hat\otimes U_\hbar\mathfrak{g}_R$. It can be rewritten as

$$\Phi = (F_2^{-1})^{23}(1\otimes\bar\Delta)(F_2^{-1})[(F_2^{-1})^{12}(\bar\Delta\otimes 1)(F_2^{-1})]^{-1};$$

it follows that $\Phi\in U_\hbar\mathfrak{g}_R\hat\otimes U_\hbar\mathfrak{g}^{\hat\otimes 2}$. Finally, Φ can also be written as

$$\Phi = [(1\otimes\Delta_R)F_1](F_2^{-1})^{23}F^{23}(\Delta\otimes 1)(F^{-1})(\Delta\otimes 1)(F_2)(F_1^{-1})^{12}.$$

But $(\Delta\otimes 1)(F^{-1})$ belongs to $U_\hbar\mathfrak{n}_+\otimes U_\hbar\mathfrak{g}_+\otimes U_\hbar\mathfrak{n}_-$, by Prop. 4.4. Therefore, Φ belongs to $U_\hbar\mathfrak{g}\otimes(U_\hbar\mathfrak{g}_R U_\hbar\mathfrak{g}_+ U_\hbar\mathfrak{g}_R)\otimes U_\hbar\mathfrak{g}$.

Φ can again be rewritten as

$$\Phi = F_2^{-1(23)}(1\otimes\bar\Delta)(F_1)(1\otimes\bar\Delta)(F^{-1})F^{(12)}(\Delta\otimes 1)(F_2)F_1^{-1(12)}.$$

Proposition 4.4 now implies that $(1 \otimes \bar{\Delta})(F_1)$ belongs to $U_\hbar\mathfrak{g} \otimes U_\hbar\mathfrak{g}_- \otimes U_\hbar\mathfrak{g}$. Therefore, Φ belongs to $U_\hbar\mathfrak{g} \otimes (U_\hbar\mathfrak{g}_R U_\hbar\mathfrak{g}_- U_\hbar\mathfrak{g}_R) \otimes U_\hbar\mathfrak{g}$.

By the PBW results of sect. 3.1, the intersection of $U_\hbar\mathfrak{g}_R U_\hbar\mathfrak{g}_+ U_\hbar\mathfrak{g}_R$ and $U_\hbar\mathfrak{g}_R U_\hbar\mathfrak{g}_- U_\hbar\mathfrak{g}_R$ is reduced to $U_\hbar\mathfrak{g}_R$. Therefore, Φ belongs to $U_\hbar\mathfrak{g} \otimes U_\hbar\mathfrak{g}_R \otimes U_\hbar\mathfrak{g}$. ∎

Let

(87) $$u_R = m(1 \otimes S)(F_1),$$

with m the multiplication of $U_\hbar\mathfrak{g}$.

THEOREM 6.1: *The algebra $U_\hbar\mathfrak{g}$, endowed with the coproduct Δ_R, associator Φ, counit ε, antipode $S_R = \mathrm{Ad}(u_R) \circ S$, respectively defined in (85), (86), (22), (23), (24), (87), and R-matrix*

(88) $$\mathcal{R}_R = [a^{21}(\Pi' \otimes 1)(F^{21})] q^{D \otimes K} q^{\frac{1}{2}\sum_{i \in \mathbb{N}} h^+[e^i] \otimes h^-[e_i]} [(\Pi \otimes 1)(F) F_{\Pi,\Pi'} a^{-1}],$$

is a quasi-triangular quasi-Hopf algebra. $U_\hbar\mathfrak{g}_R$ is a sub-quasi-Hopf algebra of it. Moreover, $\mathcal{R}_R$ belongs to $U_\hbar\mathfrak{g}_R \hat{\otimes} U_\hbar\mathfrak{g}$.

Proof: The statement on $U_\hbar\mathfrak{g}$ follows directly from [6], section 1, Remark 5. That $U_\hbar\mathfrak{g}_R$ is a sub-quasi-bialgebra of $U_\hbar\mathfrak{g}$ follows from Proposition 6.1 and Proposition 6.4. Let us now show that S_R preserves $U_\hbar\mathfrak{g}_R$.

We have $\sum_i x_i S_R(x_i') = \varepsilon(x)$, for $x \in U_\hbar\mathfrak{g}$, where $\Delta_R(x) = \sum_i x_i \otimes x_i'$. Let (m_α) be a basis of $U_\hbar\mathfrak{g}$ as a left $U_\hbar\mathfrak{g}_R$-module, with $m_0 = 1$. Set $S_R = \sum_\alpha S_R^{(\alpha)} m_\alpha$, with $S_R^{(\alpha)}$ some linear map from $U_\hbar\mathfrak{g}_R$ to itself. Then for $\alpha \neq 0$, $\sum_i x_i S_R^{(\alpha)}(x_i') = 0$. Let us show that this implies that $S_R^{(\alpha)} = 0$.

Assume that for some α, $S_R^{(\alpha)}$ is not 0. Dividing it by the largest possible power of $\hbar$, we may assume that its classical limit $S_{R,cl}^{(\alpha)}$ is non-zero. $S_{R,cl}^{(\alpha)}$ is then a map from $U\mathfrak{g}_R$ to itself, such that $\sum_i y_i S_{R,cl}^{(\alpha)}(y_i') = 0$ for any $y \in U\mathfrak{g}_R$, where $\Delta(y) = \sum_i y_i \otimes y_i'$. We then check by induction on the degree of y that $S_{R,cl}^{(\alpha)} = 0$, a contradiction.

So $S_R = S_R^{(0)}$, and S_R preserves $U_\hbar\mathfrak{g}_R$.

That $\mathcal{R}_R$ belongs to $U_\hbar\mathfrak{g}_R \hat{\otimes} U_\hbar\mathfrak{g}$ follows clearly from (88). ∎

6.3 ADELIC ALGEBRAS. In [6], Drinfeld also defined an adelic version of the Manin pair $(\mathfrak{a} \otimes k, \mathfrak{a} \otimes R)$. Let $\mathbb{A}$ be the ring of adeles of X and $\mathbb{C}(X)$ be the field of meromorphic functions of X. Let us define on $\mathbb{A}$ the scalar product $\langle f, g \rangle_{\mathbb{A}} = \sum_{x \in X} \mathrm{res}_x(fg\omega)$. Endow $\mathfrak{a} \otimes \mathbb{A}$ with the scalar product of the Killing form of $\mathfrak{a}$ with $\langle,\rangle_{\mathbb{A}}$. The Lie algebra $\mathfrak{a}\otimes\mathbb{C}(X)$ is then a Lagrangian subspace of $\mathfrak{a}\otimes\mathbb{A}$; the pair $(\mathfrak{a}\otimes\mathbb{A},\mathfrak{a}\otimes\mathbb{C}(X))$ then forms a Manin pair.

It is easy to double extend it as in section 1.2. The construction of quasi-Hopf algebras presented here then can be applied to yield an "adelic" quasi-Hopf algebra quantizing this Manin pair.

6.4 Quantum Weyl group action. The Weyl group W of $\mathfrak{a}$ naturally maps to the group of automorphisms of the Manin pair $(\mathfrak{g},\mathfrak{g}_R)$. There is an algebra automorphism w of $U_\hbar\mathfrak{g}$, deforming the action of the nontrivial element of W. It is defined by the rules

$$(w\cdot e)(z)=-\left(q^{K\partial}(q^{h^-}f)\right)(z),\quad (w\cdot f)(z)=-e(z)q^{-((T+U)h^+)(z)},$$
$$w(h^+[r])=-h^+[r],\quad w(h^-[\lambda])=-h^-[\lambda],\quad w(D)=D, w(K)=K,$$

where $r\in R,\lambda\in\Lambda$ and $(w\cdot x)(z)=\sum_{i\in\mathbb{Z}}w(x[\epsilon^i])\epsilon_i(z)$, $x=e,f$.

Note that w does not preserve $U_\hbar\mathfrak{g}_R$, and $w^2\neq 1$.

7. Analogues and generalizations of $U_\hbar\mathfrak{g}$

7.1 Replacing $q^{K\partial}$ by a general automorphism. The algebras $U_\hbar\mathfrak{g}$ presented in section 2.3 admit the following generalizations. Let σ be a ring automorphism of k, commuting with ∂ and preserving R and $\langle,\rangle_k$. Then we can form an algebra $U_\hbar\mathfrak{g}_{k,\sigma}$ with the same generators as $U_\hbar\mathfrak{g}$ (except K), replacing in all relations $q^{K\partial}$ by σ. For example, (6) becomes

$$[h^+[r],h^-[\lambda]]=\frac{2}{\hbar}\langle(1-\sigma^{-1})r,\lambda\rangle_k,$$

etc. Expressing the action of $\mathrm{Ad}(q^{KD})$ on $U_\hbar\mathfrak{g}$ and replacing $q^{K\partial}$ by σ in the resulting formulas, we obtain an (outer) automorphism Σ of $U_\hbar\mathfrak{g}_{k,\sigma}$. The formulas for Σ are

$$\Sigma(q^{-((T+U)h^+)(z)}x(z))=\sigma^{-1}(q^{-((T+U)h^+)(z)}x(z)),\quad \Sigma(D)=D,$$
$$\Sigma(h^+[r])=h^+[\sigma(r)],$$
$$\Sigma(h^-[\lambda])=h^-[(\sigma(\lambda))_\Lambda]+h^+[\sigma((T+U)\lambda)-(T+U)((\sigma(\lambda))_\Lambda)],$$

$x=e,f$; in the r.h.s. of the first formula, σ^{-1} is applied to the function part.

Note that $U_\hbar\mathfrak{g}_R$ is a subalgebra of $U_\hbar\mathfrak{g}_{k,\sigma}$, and that Σ restricts to an automorphism of $U_\hbar\mathfrak{g}_R$.

If σ is finite, then we can find some τ such that the formulas defining the action of Σ on $x(z)$ are simply $\Sigma(x(z)) = \sigma^{-1}(x(z))$, $x = e, f$. Indeed, in that case the equation

$$(\sigma \otimes \sigma)\tau - \tau + \sum_{i\in\mathbb{N}} [T(\sigma((\sigma^{-1}(e_i))_\Lambda)) - T(e_i)] \otimes e^i = 0$$

can easily be solved.

If moreover $\sigma(\Lambda) = \Lambda$, then Σ coincides with σ^{-1} also on the generating series $h^\pm(z)$.

Remark 7.1: As in Remark 1.1, the algebra $U_\hbar\mathfrak{g}_{k,\sigma}$ (without D) can be generalized with k, $\langle 1, \cdot\rangle_k$, $R \subset k$ and σ replaced by an arbitrary Frobenius algebra (k_0, θ) with a Lagrangian subalgebra and an automorphism, preserving the scalar product and the subalgebra; and the full algebra $U_\hbar\mathfrak{g}_{k,\sigma}$ can be generalized to the case where k_0 is endowed in addition with a derivation ∂_0 commuting with the automorphism, and such that $\theta \circ \partial_0 = 0$.

Remark 7.2: It seems difficult to combine the above action with the quantum Weyl group action of section 6.4 to give quantizations of more general "twisted" Manin pairs of Drinfeld. In that situation, X is endowed with an involution σ preserving ω and S, and the Manin pair is defined in the algebra $(\mathfrak{sl}_2 \otimes k)^{\mathbb{Z}/2\mathbb{Z}}$, where the action of $\mathbb{Z}/2\mathbb{Z}$ is by the tensor product of the Weyl group action with σ. The difficulty is that Σ has finite order whereas w has infinite order. If after conjugation, w could be brought to a form preserving $U_\hbar\mathfrak{g}_R$, it might happen that the procedure described here applies.

7.2 Discrete analogues.

Let us formally express the algebra relations of $U_\hbar\mathfrak{g}$, using the new generating series $K^+(z) = q^{((T+U)h^+)(z)}$ and $K^-(z) = q^{h^-(z)}$. Let us replace the expression $q^{2\sum_{i\in\mathbb{N}}((T+U)e_i)(z)e^i(w)}$ by $q(z,w)$. Using (1), we formally derive the equation

$$q(z,w)q(w,z) = 1; \tag{89}$$

let us also note $(q^{K\partial} f)(z)$ as $f(\sigma^{-1}(z))$. Then the formulas presenting $U_\hbar\mathfrak{g}$ become

$$(K^+(z), K^+(w)) = 1, \quad (K^+(z), K^-(w)) = \frac{q(z,w)}{q(z,\sigma(w))}, \tag{90}$$

$$(K^-(z), K^-(w)) = \frac{q(\sigma(z),\sigma(w))}{q(z,w)}, \tag{91}$$

$$(K^+(z), e(w)) = q(z,w), \quad (K^-(z), e(w)) = q(w,\sigma(z)), \tag{92}$$

$$(K^+(z), f(w)) = q(w,z), \quad (K^-(z), f(w)) = q(z,w), \tag{93}$$

$$(e(z), e(w)) = q(z,w), \quad (f(z), f(w)) = q(w,z), \tag{94}$$

$$[e(z), f(w)] = \delta_{z,w} K^+(z) - \delta_{z,\sigma(w)} K^-(w)^{-1}. \tag{95}$$

We used the standard notation (a,b) for the group commutator $aba^{-1}b^{-1}$; we also multiplied $f(z)$ by $\hbar$, and replaced $\delta(z,w)$ by $\delta_{z,w}$. We have also the trivial relations

$$K^{\pm}(z)K^{\pm}(z)^{-1} = K^{\pm}(z)^{-1}K^{\pm}(z) = 1. \tag{96}$$

Generators $K^{\pm}(z), K^{\pm}(z)^{-1}, e(z)$ and $f(z)$ and relations (90), (91), (92), (93), (94), (95) can be thought of as presenting a complex algebra $A(Z,\sigma,q)$ defined from the data of a discrete set Z, a map $\sigma : Z \to Z$, and a function $q : Z^2 \to \mathbb{C}^\times$, satisfying (89). It is easy to see that a basis for $A(Z,\sigma,q)$ is formed by the family $\prod_{z\in Z} e(z)^{\epsilon_z} \prod_{z\in Z} K^+(z)^{\kappa_z^+} \prod_{z\in Z} K^-(z)^{\kappa_z^-} \prod_{z\in Z} f(z)^{\eta_z}$, $\epsilon_z, \eta_z \in \mathbb{N}$, $\kappa_z^{\pm} \in \mathbb{Z}$.

The quantum Weyl group action of section 6.4 then has the following discrete analogue. Assume σ to be invertible. Then there is an automorphism w_Z of $A(Z,\sigma,q)$, defined by the formulas

$$w_Z(e(z)) = -K^-(\sigma^{-1}(z))f(\sigma^{-1}(z)), \quad w_Z(f(z)) = -e(z)K^+(z)^{-1},$$
$$w_Z(K^{\pm}(z)) = K^{\pm}(z)^{-1}.$$

In the case where $\sigma = id_Z$, the discrete analogue of the coalgebra structure of $U_\hbar\mathfrak{g}$ is then given by

$$\Delta(K^{\pm}(z)) = K^{\pm}(z) \otimes K^{\pm}(z), \quad \Delta(e(z)) = e(z) \otimes K^+(z) + 1 \otimes e(z),$$
$$\Delta(f(z)) = f(z) \otimes 1 + K^-(z)^{-1} \otimes f(z).$$

The subalgebras $A_+(Z, id_Z, q)$ and $A_-(Z, id_Z, q)$ of $A(Z, id_Z, q)$ generated by the $e(z)$ and $K^+(z)^{\pm 1}$, resp. by the $f(z)$ and $K^-(z)^{\pm 1}$ then form Hopf subalgebras of $A(Z, id_Z, q)$. If $A_+(Z, id_Z, q)$ is given the opposite coproduct, they are in duality, the pairing being defined by $\langle e(z), f(w)\rangle = \delta_{z,w}$, $\langle K^+(z)^\epsilon, K^-(w)^{\epsilon'}\rangle = q(z,w)^{\epsilon\epsilon'}$, $\epsilon, \epsilon' = 1$ or -1.

Remark 7.3: It is also natural to ask whether the formal series in $\hbar, z$ and w given by $\exp(2\hbar \sum_i ((T+U)h^+)(z)e_i(w))$ has an analytic prolongation. It could then happen that the relations defining $A(Z,\sigma,q)$ can be represented over $\mathbb{C}$ in a weak sense – as relations between analytic prolongations of matrix coefficients of operators acting on highest weight modules.

Remark 7.4: In the situation where k and R are replaced by a finite dimensional Frobenius algebra and a maximal isotropic subring of it, an expression for F is

$$F = \exp(\hbar \sum_i e[\epsilon^i] \otimes f[\epsilon_i]), \tag{97}$$

for $(\epsilon^i), (\epsilon_i)$ two dual bases of k. As it was pointed out in [4], this result is no longer true when k is infinite-dimensional. For example, such results as the commutativity of $\sum_i e[\epsilon^i] \otimes f[\epsilon_i]$ with $e(z) \otimes K^+(z) + 1 \otimes e(z)$ cease to be true in the case where (k, R) are associated to a curve with marked points. This is because, by vanishing properties of $1 + a_0\psi_+$ (see [11]), the defining relations only imply an identity

$$(z - q^{-\partial}w)[(e \otimes f)(z), (e \otimes q^{(T+U)h^+})(w)] = 0,$$

so that

$$[(e \otimes f)(z), (e \otimes q^{(T+U)h^+})(w)] = A(z)\delta(z, q^{-\partial}w),$$

for some field $A(z)$, so that $[\sum_i e[\epsilon^i] \otimes f[\epsilon_i], (e \otimes q^{(T+U)h^+})(w)] = A(q^{-\partial}w)$.

In [4], an expression for F, well-defined up to order 2, was proposed in the quantum affine case and checked up to that order.

It would be interesting to obtain expressions for F is the framework of [13]; one could expect to check their coincidence with those of [14].

An earlier version of the present work used a (wrong) generalization of (97) to the infinite-dimensional setting; after that, the works [8, 9] were completed, relying on this work. However, after F is defined by Definition 5.1, all results of that version are correct, except those involving commutation relations of $\sum_i e[\epsilon^i] \otimes f[\epsilon_i]$. This shows that the only corrections to [8, 9] are to replace the definition of F based on the generalization of (97) by Definition 5.1.

8. Appendix: maximal isotropy of rings

Let $\mathbb{A}$ be the adeles ring of X, and $\mathbb{C}(X)$ be the field of functions over X. Define on $\mathbb{A}$ the bilinear pairing $\langle , \rangle_{\mathbb{A}}$ by

$$\langle f, g\rangle_{\mathbb{A}} = \sum_{x \in X} \mathrm{res}_x(fg\omega).$$

In [6], Drinfeld made the following statement.

LEMMA 8.1: *$\mathbb{C}(X)$ is maximal isotropic in $\mathbb{A}$ w.r.t. $\langle , \rangle_{\mathbb{A}}$.*

Below we prove this and the similar statement

LEMMA 8.2: *R is maximal isotropic in k w.r.t. $\langle , \rangle_k$.*

Recall first the duality theorem ([15], II-8, Theorem 2). Let D be any divisor on X, and $\Omega(D)$ be the space of all meromorphic forms ω equal to zero or such that their divisor is $\geq D$. Let, on the other hand, $\mathbb{A}_{\geq -D}$ be the space of adeles with divisor $\geq -D$. Then $\langle , \rangle_{\mathbb{A}}$ induces a non-degenerate pairing

$$\Omega(D) \times (\mathbb{A}/(\mathbb{A}_{\geq -D} + \mathbb{C}(X))) \to \mathbb{C}.$$

Proof of Lemma 8.1: The isotropy of $\mathbb{C}(X)$ follows from the residue formula. Let Ω be the space of all meromorphic forms on X, and let us now show that the pairing

$$\Omega \times (\mathbb{A}/\mathbb{C}(X)) \to \mathbb{C} \tag{98}$$

is also non-degenerate. Let $f \in \mathbb{A}/\mathbb{C}(X)$ have vanishing pairing with Ω. Then for any divisor D, the pairing of its image in $\mathbb{A}/(\mathbb{A}_{\geq -D} + \mathbb{C}(X))$ with any element of $\Omega(D)$ is zero, which means that f belongs to $\mathbb{A}_{\geq -D}/(\mathbb{A}_{\geq -D} \cap \mathbb{C}(X))$ for any D, and so is zero.

The lemma now follows from the non-degeneracy of (98). ∎

Proof of Lemma 8.2: Let for any divisor $\bar{D}$ with support in S, $k_{\geq \bar{D}}$ be the space of elements of k with divisor $\geq \bar{D}$.

LEMMA 8.3: *Let D_0 be a divisor of X, supported in S. Then the mappings $\mathbb{A} \to k$ induces an isomorphism of $\mathbb{A}/(\mathbb{A}_{\geq -D_0} + \mathbb{C}(X))$ with $k/(k_{\geq -D_0} + R)$.*

Proof: Let $D = n(\sum_{s \in S} s)$. For n large enough, $D \geq (\omega_0)$ and $\Omega(D) = 0$; by the duality theorem, this implies that $\mathbb{A}/(\mathbb{A}_{\geq -D} + \mathbb{C}(X)) = 0$, and so $\mathbb{A} = \mathbb{A}_{\geq -D} + \mathbb{C}(X)$. Let D_0 be a divisor $\leq D$, then $\mathbb{A}_{\geq -D_0} \subset \mathbb{A}_{\geq -D}$, so

$$\begin{aligned}\mathbb{A}/(\mathbb{A}_{\geq -D_0} + \mathbb{C}(X)) &= (\mathbb{A}_{\geq -D} + \mathbb{C}(X))/(\mathbb{A}_{\geq -D_0} + \mathbb{C}(X)) \\ &= \mathbb{A}_{\geq -D}/(\mathbb{A}_{\geq -D_0} + (\mathbb{C}(X) \cap \mathbb{A}_{\geq -D})).\end{aligned}$$

Let for any n, $Q_n = \mathbb{A}_{\geq -n(\sum_{s \in S} s)}/\left(\mathbb{A}_{\geq -D_0} + (\mathbb{C}(X) \cap \mathbb{A}_{\geq -n(\sum_{s \in S} s)})\right)$. For the same n as above, the natural maps $Q_n \to Q_{n+1}$ are isomorphisms. On the other hand, we have a map $Q_n \to k/(k_{\geq -D_0} + R)$; its kernel is the set of adeles $\geq -n(\sum_{s \in S} s)$, $\geq -D_0$, and in R, so is zero. So $Q_n \to k/(k_{\geq -D_0} + R)$ is injective; it is also surjective, as we can see by composing it with a suitable $Q_n \to Q_m$, m large enough, so it is an isomorphism. ∎

From the duality theorem it now follows that for D_0 like in Lemma 7.3, the residue pairing

$$\Omega(D_0) \times (k/(k_{\geq -D_0} + R)) \to \mathbb{C}$$

is non-degenerate. Let us specialize D_0 to $D_{0,m} = -m(\sum_{s\in S} s)$; then the natural maps induce an inductive system $\Omega_{0,m} \subset \Omega_{0,m'}$ and a projective system $k/(k_{\geq -D_{0,m'}} + R) \to k/(k_{\geq -D_{0,m}} + R)$, $m \leq m'$, compatible with the duality. It follows that the induced pairing between the inductive and projective limits is non-degenerate. Since $\bigcup_{m\geq 0} \Omega(D_{0,m})$ is the set of forms on X regular outside S, and ω has neither zeros nor poles outside S, this space is equal to ωR. On the other hand, $\lim_{m\geq 0} k/(k_{\geq -D_{0,m}} + R) = k/R$. We conclude that the pairing

$$\omega R \times (k/R) \to \mathbb{C}$$

induced by the residue is non-degenerate, whence the lemma. ∎

Remark 8.1: Another proof of Lemma 7.2 can be obtained following [6], section 2.

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